Kinematics And Dynamics Of Machines

(K & DM)

Semester : 4TH

Branch : Mechanical Engineering

Module - III

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The concept of velocity and acceleration images is used extensively in the kinematic analysis of mechanisms having ternary, quaternary, and higher- order links. If the velocities and accelerations of any two points on a link are known, then, with the help of images the velocity and acceleration of any other point on the link can be easily determined. An example is

1. Instantaneous Centre Method
2. Relative Velocity Method
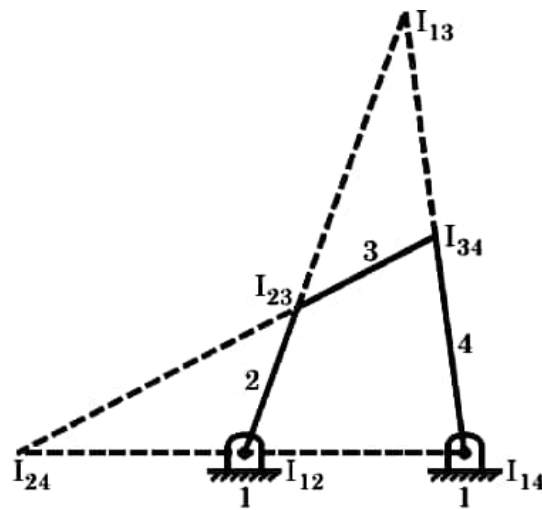
Velocity by Instantaneous Centre Method

Instantaneous centre is one point about which the body has pure rotation. Hence for the body which having straight line motion, the radius of curvature of it is at infinity and hence instantaneous centre of this ties at infinite.

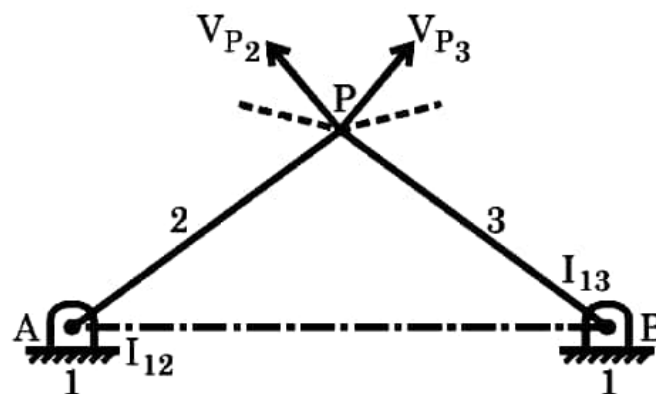## Special cases of ICR
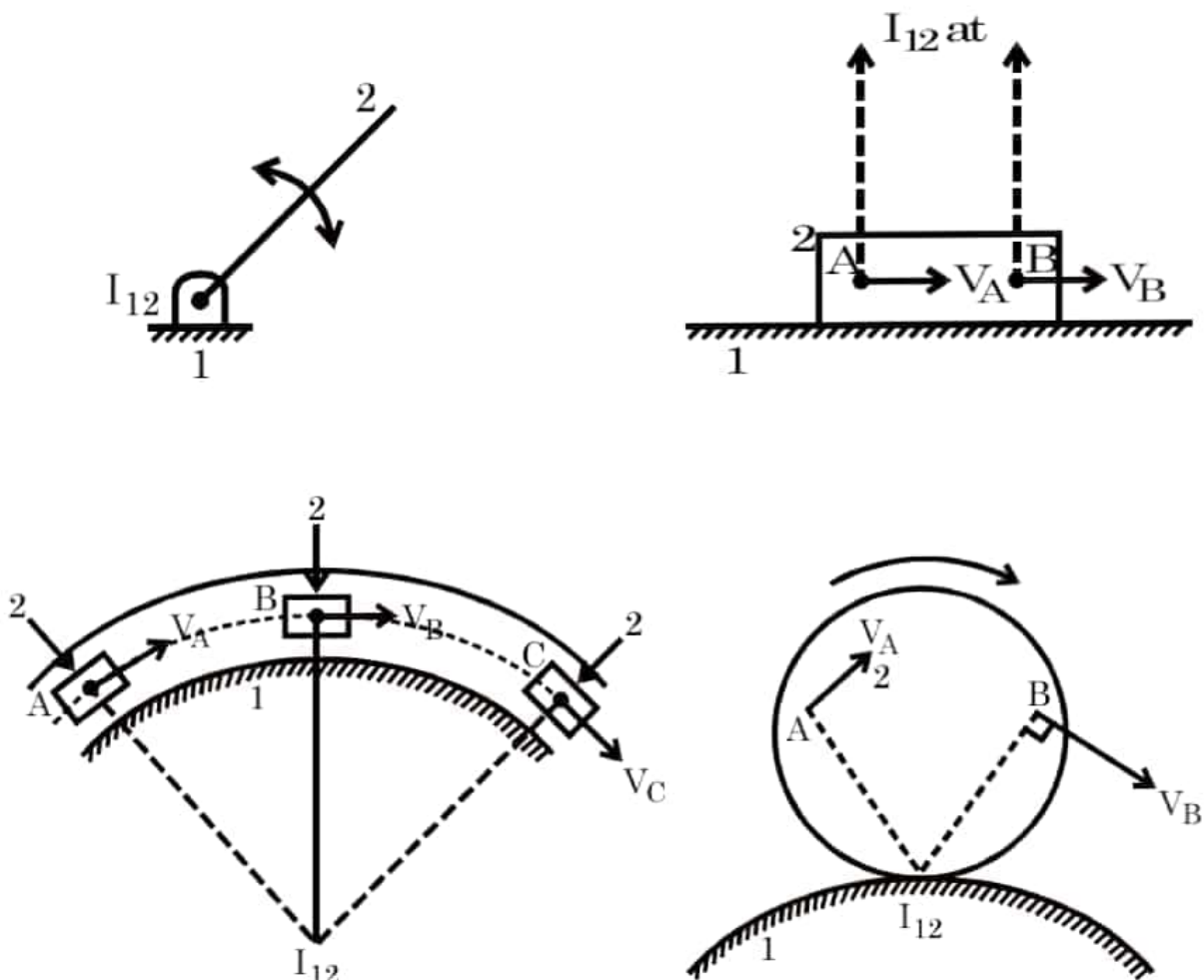
Types of ICR:

(i)   Fixed ICR: I_{12}, I_{14}
(ii)   Permanent ICR: I_{23}, I_{34}
(iii)   Neither Fixed nor Permanent I.C: I_{13}, I_{24}

Three-Centre-in-line Theorem (Kennedy's Theorem)

Kennedy Theorem states that "If three links have relative motion with respect to each other, their relative instantaneous centre lies on straight line".

The Theorem can be proved by contradiction.

The Kennedy Theorem states that the three IC I_{12}, I_{13}, I_{23} must all lie on the same straight line on the line connecting two pins.

Let us suppose this is not true and I_{23} is located at the point P. Then the velocity of P as a point on link 2 must have the direction V_{P_2}, $\perp$ to AP. Also the velocity of P as a point on link 3 must have the direction V_{P_2}, $\perp$ to BP. The direction is inconsistent with the definition that an instantaneous centre must have equal absolute velocity as a part of either link. The point P chosen therefore, cannot be the IC I_{23}.

This same contradiction in the direction of V_{P_2} and V_{P_3} occurs for any location chosen for point P, except the position of P chosen on the straight line passing through I_{12} and I_{13}. This justify the Kennedy Theorem.

Properties of the IC:

1. A rigid link rotates instantaneously relative to another link at the instantaneously centre for the configuration of the mechanism considered.

2. The two rigid links have no linear velocity relative to each other at the instantaneous centre. In other words, the velocity of the IC relative to any third rigid link will be same whether the instantaneous centre is regarded as a point on the first rigid link or on the second rigid link.

## Number of I.C in a mechanism:

$$N = \frac{n(n-1)}{2}$$

N = no. of I.C.
n = no. of links.

1. Each configuration of the link has one centre.
   The instantaneous centre changes with alteration of configuration of mechanism.

Method of locating instantaneous centre in mechanism

Consider a pin jointed four bar mechanism as shown in fig. The following procedure is adopted for locating instantaneous centre.
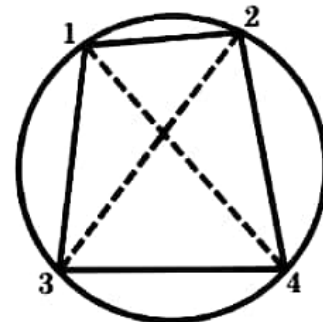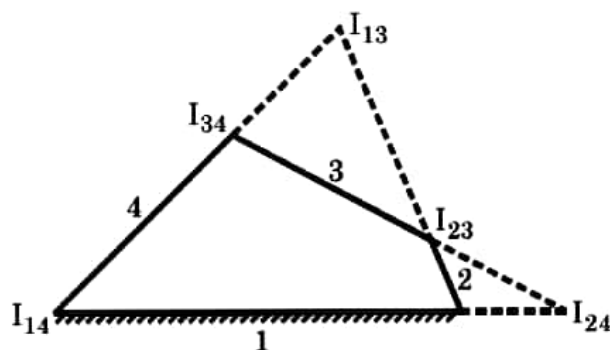
1. First of all, determine the no. of IC.
   $$N = \frac{n(4-1)}{3} = \frac{4(4-1)}{2} = 6$$

2. Make a list at all the instantaneous centre in a mechanism.

Links	1	2	3	4
–	12	23	34	–
IC	13	24		
	14			

3. Locate the fixed and permanent instantaneous centre by inspection. In fig I_{12} and I_{14} are fixed I.Cs and I_{23} and I_{34} are permanent instantaneous centre locate the remaining neither fixed nor permanent IC by Kennedy's Theorem. This is done by circle diagram

as shown mark the points on a circle equal to the no. of links in mechanism. In present case 4 links.

4. Join the points by solid line to show these centres are already found. In the circle diagram these lines are 12, 23, 34, and 14 to indicate the ICs I_{12}, I_{23}, I_{34} and I_{14},

5. In order to find the other two IC, join two such points that the line joining them forms two adjacent triangles in the circle diagram. The line which is responsible for completing two triangles should be a common side to the two triangles. In fig join 1 and 3 to form triangle 123 and 341 and the instantaneous centre I_{13} will lie on the intersection of I_{12}, I_{23} and I_{14} I_{34}. similarly IC I_{24} is located.

Angular Velocity Ratio Theorem

According to this Theorem "the ratio of angular velocity of any two links moving in a constrained system is inversely proportional to the ratio of distance of their common instantaneous centre from their centre of rotation".
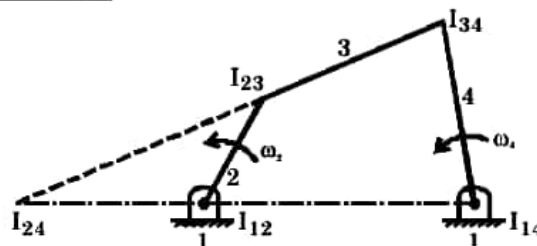
$$\frac{\omega_2}{\omega_3} = \frac{I_{13}\,I_{23}}{I_{12}\,I_{23}}$$

$$\frac{\omega_2}{\omega_4} = \frac{I_{14}\,I_{24}}{I_{12}\,I_{24}}$$

Indices of Merit (Mechanical Advantage)

From previous concept are know that

$$\boxed{\frac{\omega_2}{\omega_4} = \frac{I_{14}\,I_{24}}{I_{12}\,I_{24}}}$$ as per angular velocity ratio Theorem.

Let T_2 represent the input torque T_4 represent the output torque. Also consider that there is no friction or inertia force.

Then $T_2\omega_2 = T_4\omega_4$

– ve sign indicates that power is applied to link 2 which is negative of the power applied to link 4 by load.

$$\therefore \qquad \boxed{\frac{T_4}{T_2} = \frac{\omega_2}{\omega_4} = \frac{I_{14}\,I_{24}}{I_{12}\,I_{24}}}$$

The mechanical advantage of a mechanism is the instantaneous ratio of the output force (torque) to the input force (torque). From above equation we know that mechanical advantage is the reciprocal of the velocity ratio.

Fig shows a typical position of four bar linkage in toggle, where link 2 and 3 are on the same straight line.
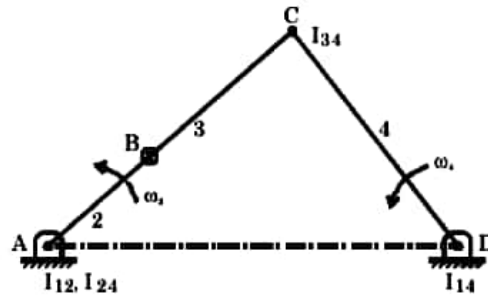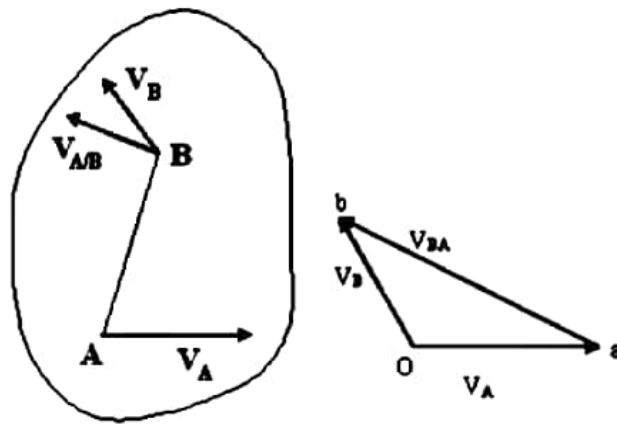
At this position, I_{12} and I_{24} is coincident at A and hence, the distance $I_{24} I_{24}$ is zero,

$$\therefore \quad \frac{\omega_4}{\omega_2} = \frac{I_{12} I_{24}}{I_{14}, I_{24}} = \frac{\textcircled{0}}{I_{14} I_{24}} = 0$$

$$\therefore \qquad \boxed{\omega_4 = 0}$$

$\therefore$ Mechanical advantage $\dfrac{T_4}{T_2} = \dfrac{\omega_2}{\omega_4} = \infty$

Hence the mechanical advantage for the toggle position is infinity


The relative velocity method is based upon the velocity of the various points of the link. Consider two points A and B on a link. Let the absolute velocity of the point A i.e. VA is known in magnitude and direction and the absolute velocity of the point B i.e. VB is known in direction only. Then the velocity of B may be determined by drawing the velocity diagram as shown.

1. Take some convenient point o, Known as the pole.
2. Through o, draw oa parallel and equal to VA, to some convenient scale.
3. Through a, draw a line perpendicular to AB. This line will represent the velocity of B with respect to A, i.e.
4. Through o, draw a line parallel to VB intersecting the line of VBA at b.
5. Measure ob, which gives the required velocity of point B to the scale.

1. Relative Velocity and Acceleration:

Relative velocity of coincident points in two kinematic links:
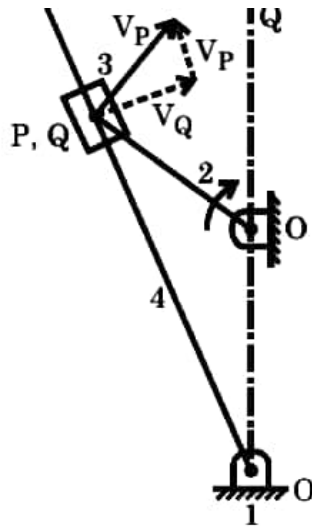
P on link 2 and 3
Q on link 4

Rubbing Velocity:

Let r_b = radius of pin B.

$\omega_{2/3}$ = relative angular velocity between link 2 and 3.

Crank pin B
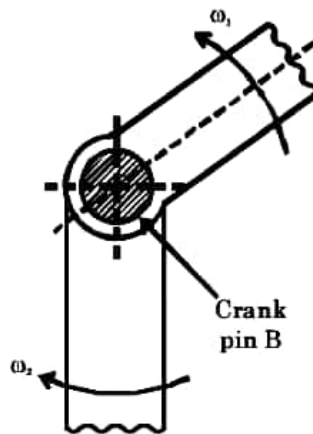
$$\boxed{\upsilon_{rub} = r_b . \omega_{2/3}}$$

$\omega_{2/3} = \omega_2 \pm \omega_3$, + for opposite rotation.

Relative Acceleration Method:

$$\boxed{f = f_c + f_t}$$

f = total acceleration

f_c = Centripetal acceleration

f_t = Tangential acceleration

$$f_c = r\omega^2 = \frac{V^2}{r} \qquad f_t = r\alpha$$

Where,

r = radius of rotation of a point on link

ω = Angular velocity of rotation

V = linear velocity of a point on link

α = Angular acceleration

Direction of f_c is along radius of rotation and towards centre.

Direction of f_t is perpendicular to radius of rotation.
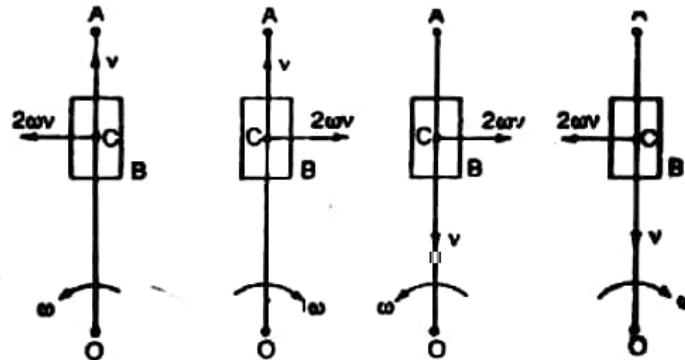
Corioli's Component of Acceleration:

Fig. Direction of coriolis component of acceleration

This tangential component of acceleration of the slider B with respect to the coincident point C on link is known as coriolis component of acceleration and is always perpendicular to the link.

∴ Coriolis component of the acceleration of B with respect to C,

$$a_{BC}^c = a_{BC}^t = 2\omega.v$$

Where
ω = Angular velocity of the link OA, and

V = Velocity of slider B with respect to coincident point C.

Pantograph

A pantograph is an instrument used to reproduce to an enlarged or a reduced scale and as exactly as possible the path described by a given point.

(i)
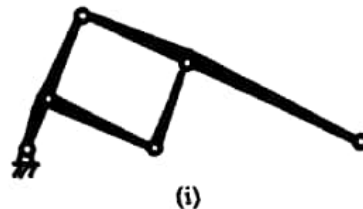
A pantograph is mostly used for the reproduction of plane areas and figures such as maps, plans etc., on enlarged or reduced scales. It is, sometimes, used as an indicator rig in order to reproduce to a small scale the displacement of the crosshead and therefore of the piston of a reciprocating steam engine. It is also used to guide cutting tools.

Velocity and Acceleration by Analytical method

Kinematic analysis of piston in I.C engine:
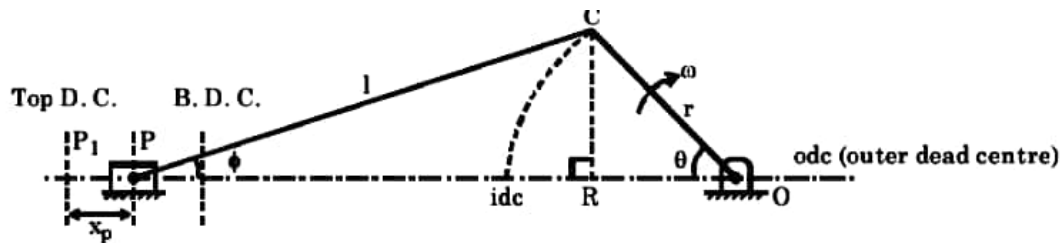
Let, r = length of crank

l = length of connecting rod

n = obliquity ratio

$$= \frac{l}{r}$$

ω = angular velocity of the crank

θ = inclination of the crank to i.d.c.

ϕ = inclination of connecting rod to the line of stroke.

x_P = displacement of piston

V_P = velocity of the piston

f_P = acceleration of the piston.

When the crank rotates through angle θ from its inner dead centre position the piston, receives displacement x_P.

$\therefore$ Displacement $x_P = P_1 P$

$$= P_1 O = PO$$

$$= (l + r) - (l\cos\phi + r\cos\theta)$$

$$x_P = r(1 - \cos\theta) + l(1 - \cos\phi)$$

Now from figure

$$CR = r\sin\theta - l\sin\phi$$

$$\sin\theta = \frac{r}{l}\sin\theta = \frac{\sin\theta}{n}$$

$$\cos\phi = \sqrt{1 - \sin^2\phi} = \sqrt{1 - \left(\frac{\sin\theta}{n}\right)^2}$$

$$= \sqrt{\frac{n^2 - \sin^2\theta}{n^2}} = \frac{\sqrt{n^2 - \sin^2\theta}}{n}$$

$\therefore$

$$x_P \approx r(1 - \cos\theta) + l\left[1 - \frac{\sqrt{n^2 - \sin^2\theta}}{n}\right]$$

$$= r(1 - \cos\theta) + \frac{l}{n}(n - \sqrt{n^2 - \sin^2\theta})$$

$$= r[1 - \cos\theta + n - \sqrt{n^2 - \sin^2\theta}]$$

$\therefore$

$$\boxed{x_P = r[1 - \cos\theta + n - \sqrt{n^2 - \sin^2\theta}]}$$

Now differentiating above equation with respect to 't'.

$$V_P = \frac{d}{dt}x_P = \frac{d}{dt}[r(1 - \cos\theta + n - \sqrt{n^2 - \sin^2\theta})]$$

$$= \frac{d}{d\theta}[r(1 - \cos\theta + n - \sqrt{n^2 - \sin^2\theta})].\frac{d\theta}{dt}$$

$$= r\omega \cdot \frac{d}{d\theta}[1 - \cos\theta + n - \sqrt{n^2 - \sin^2\theta}]$$

$$\boxed{V_p = r\omega\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right] \approx r\omega\left(\sin\theta + \frac{\sin 2\theta}{2n}\right)}$$

Again differentiating above equation w.r.to t

$$\boxed{f_p = r\omega^2\left[\cos\theta + \frac{\cos 2\theta}{n}\right]}$$

Analysis of connecting rod in I.C. engine:

Let, ω_r = angular velocity of the C.R.

α_r = angular velocity of the C.R

now $CR = r\sin\theta = l\sin\phi$

$$= \sin\phi = \frac{\sin\theta}{n}$$

now Differentiating above w.r. to 't'

$$\frac{d}{dt}\sin\phi = \frac{1}{n}\frac{d}{dt}\sin\theta$$

$$\frac{d}{d\phi}\sin\phi \cdot \frac{d\phi}{dt} = \frac{1}{n}\frac{d}{d\theta}\sin\theta \cdot \frac{d\theta}{dt}$$

$$\cos\phi \cdot \omega_r = \frac{\cos\theta}{n} \cdot \omega$$

$$\omega_r = \frac{\omega\cos\theta}{n\cos\phi}$$

Also from previous section

$$\cos\phi = \sqrt{\frac{n^2 - \sin^2\theta}{n^2}}$$

$$\therefore \omega_r = \frac{\omega\cos\theta}{n \cdot \dfrac{\sqrt{n^2 - \sin^2\theta}}{n^2}} \approx \frac{\omega\cos\theta}{n}$$

$$\alpha_r = \frac{-\omega^2\sin\theta}{n}$$

Q.   In a slider crank mechanism the stroke of the slider is 200 mm and the obliquity ratio is 4.5. The crank rotates uniformly at 400 rpm clockwise. While the crank is approaching the inner dead center and the connecting rod is normally to the crank. Find
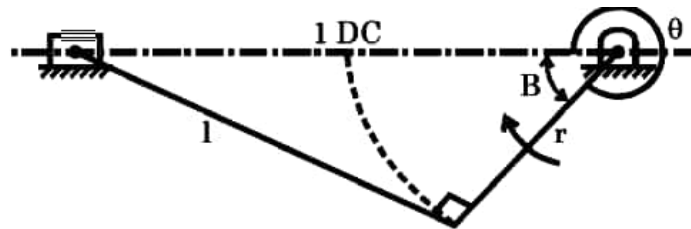(i) Velocity of piston and angular velocity of the C.R.
(ii) Acceleration of the piston and angular acceleration of the C.R.

Solution:
Stroke length = 200 mm = 2r
n = 4.5, N = 400 rpm, clockwise.
There find V_P, f_P, ω_r, α_r

Now r = 100 mm

n = 4.5

∴ l = 450 mm

ω = 41.88 rad/sec. clockwise

$\beta = \tan^{-1}\dfrac{l}{r} = 77.47°$

∴ θ = 360 − 77.47 = 282.53°

(i) $V_P = r\omega\left(\sin\theta + \dfrac{\sin 2\theta}{2n}\right)$

$= 100 \times 41.88\left(\sin 282.53 + \dfrac{\sin 565.05}{9}\right)$

$= -4286.17$ mm/sec

$= -4.29$ m/sec (direction away from the crank)

(ii) Acceleration

$f_p = r\omega^2\left(\cos\theta + \dfrac{\cos 2\theta}{n}\right)$

$= 100 \times 41.88^2\left(\cos 282.53 + \dfrac{\cos 565.05}{4.5}\right)$

$= 2.75$ m/sec² (direction towards the crank)

(iii) $\omega_r = \dfrac{\omega\cos\theta}{n} = 2.02$ rad/sec.

(Direction of ω_r is opposite to that of ω)

(iv) $\alpha_r = \dfrac{-\omega^2\sin\theta}{n} = -380.63$ rad/sec².

(Direction of α_r is same as that of ω).

Q. In an I.C engine mechanism having obliquity ratio n, show that for uniform engine speed the ratio for piston acceleration at the beginning of stroke and end of the stroke is given by $\dfrac{1+n}{1-n}$.

Solution:

P_1 is beginning of stroke

P_2 is end of stroke

at $P_1, \theta = 0$

and at $P_1\ \theta = 180°$

$$f_P = r\omega^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$f_{P_1} = r\omega^2\left(1 + \frac{1}{n}\right) = r\omega^2\left(\frac{1+n}{n}\right)$$

$$f_{P_2} = r\omega^2\left(-1 + \frac{1}{n}\right) = r\omega^2\left(\frac{1-n}{n}\right)$$

$$\therefore \qquad \boxed{\frac{f_{P_1}}{f_{P_2}} = \frac{1+n}{1-n}}.$$

Velocity and Acceleration by Klein's Construction:

OC — crank

CP — connecting rod

θ — Angle made by crank with i.d.c

ω — Angular velocity of crank.

Procedure to draw velocity diagram:

1. Firstly draw the configuration diagram of slider crank mechanism to the scale 1: K.
2. After getting configuration diagram OCP, now draw a line through 'O' ⊥ to the line of stroke OP.
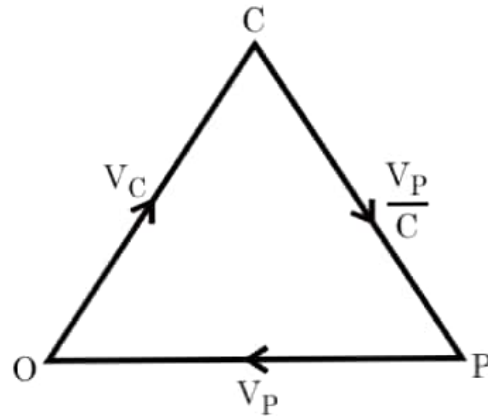
3. Extend the connecting rod length PC, to meet this ⊥ name the intersection point as M.
4. $\triangle$ OCM Represent the velocity polygon of slider crank mechanism to the scale 'KW'.

   In $\triangle$ OCM,

   OC represents V_c,

   CM represents $V_{P/C}$

   And OM represents V_P

$$\therefore \quad \boxed{V_c = OC.K\omega}$$

$$\boxed{V_{P/C} = OM.K\omega}$$

$$\boxed{V_P = OM.K\omega}$$

Velocity of any point x lying on connecting rod:

$$\frac{Cx}{CP} = \frac{Cx}{CM} \Rightarrow Cx = \frac{Cx}{CP} \times CM$$

$$\therefore \quad \boxed{V_x = Ox . K\omega}$$

Acceleration Analysis:

1. As discussed earlier $\triangle$ OCM is velocity polygon of slider crank mechanism to scale 'kW'.
2. With C as centre and with CM as radius draw a circle.
3. Draw another circle with diameter as length PC.
4. Then KL represents the common chord of these two circles.
5. Extend KL to meet the line of stroke at N. Also KL is intersecting to PC at Q.
6. Then $\triangle$ OCM represent the acceleration polygon.

Special Cases of Klein's Construction:

When the crank is at i.d.c.

$OM = 0,$ $\therefore V_P = OM \times k\omega = 0$

$V_C = OC \times k\omega,$ $V_{P/C} = CM \times k\omega$ $[\because CM = OC$

$\therefore$ $\boxed{V_C = V_{P/C}}$

Acceleration polygon is OCQN.

Here $QN = 0$

$\therefore$ $f^t_{P/C} = NQ \times kW^2 = 0$

i.e., $\alpha_{P/C} = \dfrac{f^t_{P/C}}{PC} = 0$

it means angular velocity of connecting rod is maximum.

$f_c = CO \times k\omega^2$

$f_p = NO \times k\omega^2$

$f_{P/C} = QC \times k\omega^2$

(ii)   When the crank is at 90° from idc.

OCM → Velocity triangle

OCQN → Acceleration polygon.

$V_C = OC \times k\omega$

$V_{P/C} = CM \times k\omega = 0$

$\therefore$ $W_{P/C} = \dfrac{V_{P/C}}{PC} = 0$

$\therefore$ at $\theta = 90°$ $W_{P/C} = 0$

$V_P = PM \times k\omega$

$\therefore$ $\boxed{V_P = V_C}$

$f^t_{P/C} = QC \cdot k\omega^2 = 0$

$f_c = CO \times k\,\omega^2$

$f_p = -NO \times k\omega^2 \rightarrow$ case of retardation

$f^t_{P/C} = NQ \cdot k\omega^2$

When the crank is at 180° from i.d.c.
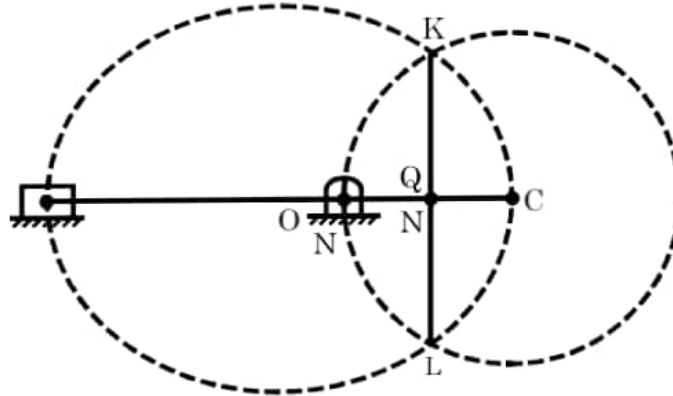
OCM $\rightarrow$ Velocity triangle

OCQN $\rightarrow$ Acceleration polygon.

$V_P = (OM) k\omega = 0$

$V_C = (OC) \cdot k\omega$

$V_{P/C} = CM \cdot k\omega \qquad\qquad V_C = V_{P/C}$

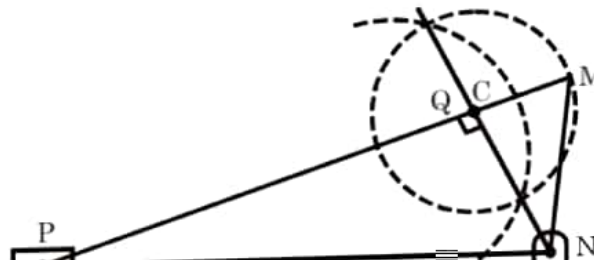$f^t_{P/C} = NQ \cdot k\omega^2 = 0 \qquad \therefore \alpha_{P/C} = 0$

i.e., it means angular velocity of C.R. is maximum

$f_C = (CO) k\omega^2$

$f_P = -(NO) k\omega^2 \rightarrow$ Retardation

$f^e_{P/C} = (QC) k\omega^2$

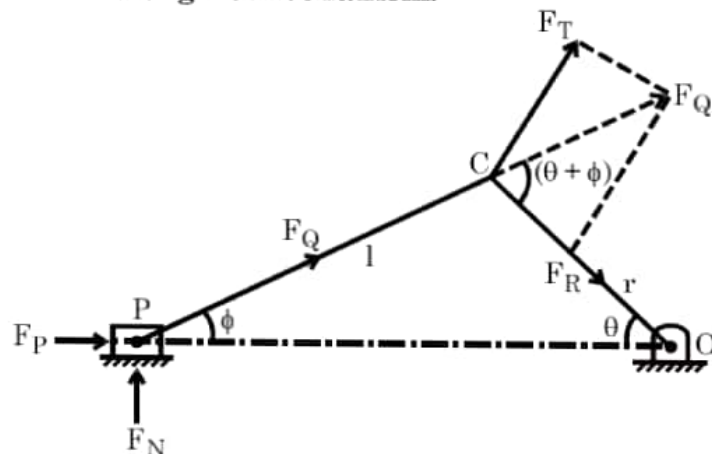(v)   When crank and connecting rod are mutual perpendicular.

$$f_{P/C} = NC. \text{K} \, \omega^2 \text{ direction N to C.}$$

Acceleration of any point X lying on connecting rod:

$$\frac{Cx}{CP} = \frac{Cx_1}{CN} \quad \Rightarrow \quad Cx_1 = \frac{Cx}{CP}.CN$$

$$\therefore \quad \boxed{f_X = Ox_1 \, . \, k\omega^2}$$

Force analysis in I.C engine Mechanism:

F_P = Net axial force on the piston or piston effort.

F_Q = Force acting along connecting rod

F_N = Normal reaction on the side of the cylinder or piston side thrust.

F_T = Tangential force at crank pin or force perpendicular to the crank.

F_R = Radial load on the crank shaft bearing.

T = Turning moment or Torque on the crank

$$\boxed{T = F_T \, . \, r}$$

*Net axial force (F_P) : During acceleration

$$\therefore \qquad \boxed{F_P = F_g - F_1}$$

$$\boxed{F_P = \frac{\pi}{4}d^2P_g - mr\omega^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right)}$$

or Different in the intensity of pressure on two sides of the piston.

F_I = inertia force = mf_p

*In case of vertical engine, the wt of the reciprocating parts (W_R) and friction force (F_f).

$$\boxed{F_P = F_g \pm F_I - F_f + W_R}$$

Q. The mass of reciprocating parts of a steam engine is 225 kg, diagram of the cylinder is 400 mm, length of the stroke is 500 mm and the ratio of length of connecting rod to crank is 4.2. When the crank is at inner dead center, the difference in the pressure of the two sides of the piston is 5 bar. At what speed must the engine run so that the thrust in the connecting rod in this piston is equal to 5200 N?

Solution: Data

m = 225 kg.

d = 400 mm = 0.4 m

Stroke length = 500 mm = 2r

∴   r = 250 mm = 0.25 m

n = 4.2

$\theta = 0°$

ρ_g = 5 bar

F_Q = 5200 N

ω =?

Here $F_Q = F_P$,

Now $F_P = F_g - F_I$

$$= \frac{\pi}{4} d^2 P_g - mr\omega^2 \left(\cos\theta - \frac{\cos 2\theta}{n} \right)$$

$$5200 = \frac{\pi}{4}(0.4)^2 \times 5 \times 10^5 - 225 \times 0.25 \times \omega^2 \left(1 + \frac{1}{4.2} \right)$$

∴ ω = 28.767 rad/sec.

N = 274.7 rpm.

Q. A single cylinder two stroke vertical engine a bore of 30 cm and a stroke of 40 cm with a connecting rod of 80 cm long. The mass of the reciprocating parts is 120 kg. When the piston is at quarter stroke and moving down, the pressure on it is 70 N/cm². If the speed of the engine crank shaft is 250 rpm clockwise find the turning moment on the crank shaft. Neglect the mass and inertia effects on connecting rods and crank.

Solution: Data

d = 30 cm

Stroke = 40 cm = 2r

∴ r = 20 cm   n = 4

  l = 80 cm

  m = 120 kg

  x_p = 0.25 × stroke length

  P_g = 70 N/cm²

  N = 250 rpm (clockwise), ω = 26.166

  T =?

Now $x_P = r(1 - \cos\theta + n - \sqrt{n^2 - \sin^2\theta})$

$0.25 \times 2r = r(1 - \cos\theta + 4 - \sqrt{16 - \sin^2\theta})$

$\dfrac{1}{2} = 5 - \cos\theta - \sqrt{16 - \sin^2\theta}$

$\sqrt{16 - \sin^2\theta} = 4.5 - \cos\theta$

$16 - \sin^2\theta = 20.25 + \cos^2\theta - 9\cos\theta$

$9\cos\theta = 20.25 - 16 + \sin^2\theta + \cos^2\theta$

$\qquad = 5.25$

$\cos\theta = \dfrac{5.25}{9}$

$\therefore\ \theta = 54.31°$
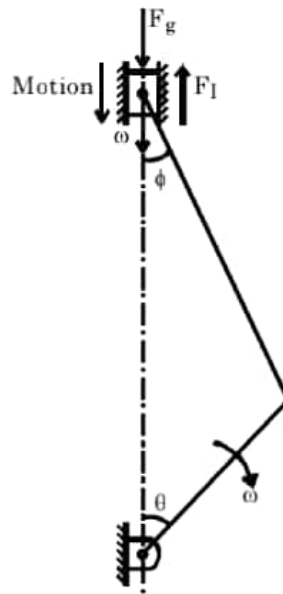
Now $\sin\phi = \dfrac{\sin\theta}{n}$

$\qquad \phi = 11.71°$

Now $F_P = F_G - F_I + \omega$

$\qquad F_G = \dfrac{\pi}{4}d^2 \cdot P_g$

$\qquad\quad = \dfrac{\pi}{4}30^2 \times 70$

$\qquad\quad = 49455\ N$

$F_I = mr\omega^2\left(\cos\theta + \dfrac{\cos 2\theta}{n}\right)$

$\quad = 120 \times 0.20 \times 26.166^2\left(\cos 54.31 + \dfrac{\cos 108.62}{4}\right)$

$\quad = 8274.7\ N$

$W = mg = 1177.2\ N$

$\therefore\ F_P = 49455 - 8274.7 + 1177.2$

$\qquad\quad = 42357.7\ N$