Lecture Note

Finite Element Methods

[FEM]

Module - I

[Unit - 1]

Semester : 5th

Branch : Mechanical Engg.

- ✓ By using energy principles, ***Argyris and Kelsey*** developed matrix structural analysis methods in 1954. This development illustrated the important role that energy principles would play in the finite element method.
- ✓ The term finite element was first introduced by ***Clough*** in 1960 in the plane stress analysis and he used both triangular and rectangular elements in that analysis.
- ✓ Most of the finite element work upto early 1960s dealt with small strains and small displacements, elastic material behaviour and static loadings. In 1961, ***Turner*** considered large deflection and thermal analysis problems. In 1962, ***Gallagher*** introduced material non-linearities problems, whereas buckling problems were initially treated by ***Gallagher and Padlog*** in 1963. In 1968, ***Zinkiewicz*** extended the method to visco elasticity problems.
- ✓ Weighted residual methods was first introduced by ***Szabo*** and ***Lee*** in 1969 for structural analysis and then by ***Zinkiewicz*** and ***Parekh*** in 1970 for transient field problems.
- ✓ During the decades of the 1960s and 1970s, the finite element method was extended to applications in shell bending, plate bending, heat transfer analysis, fluid flow analysis and general three dimensional problems in structural analysis.
- ✓ From the early 1950s to present, enormous advances have been made in the application of finite element method to solve complicated engineering problems. It is curious to note that the present day finite element method does not have its root in one discipline. The mathematicians continue to put the finite element method on sound theoretical ground whereas the engineers continue to find interesting extensions in various branches of engineering. These concurrent developments have made the finite element method as one of the most powerful approximate methods.

1.3. GENERAL STEPS OF THE FINITE ELEMENT ANALYSIS

- ✓ This section presents the general procedure of finite element analysis. For simplicity's sake, we will consider only the structural problems.
- ✓ The following two general methods are associated with the finite element analysis. They are:
  - (*i*) Force method.
  - (*ii*) Displacement or stiffness method.
- ✓ In force method, internal forces are considered as the unknowns of the problem. In displacement or stiffness method, displacements of the nodes are considered as the unknowns of the problem.
- ✓ Among these two approaches, displacement method is more desirable because its formulation is simpler for most structural analysis problems. So, a vast majority of general purpose finite element programs have used the displacement formulation for solving structural problems.

- ✓ We now present the steps along with explanations used in the finite element method formulation.

Step 1: Discretization of Structure

The art of subdividing a structure into a convenient number of smaller elements is known as discretization.

Smaller elements are classified as follows:

(*i*) One dimensional elements. (*ii*) Two dimensional elements.
(*iii*) Three dimensional elements. (*iv*) Axisymmetric elements.

(i) One dimensional elements: A bar and beam elements are considered as one dimensional elements. The simplest line element also known as linear element has two nodes, one at each end as shown in Fig.1.2.

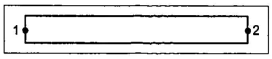


Fig. 1.2. Bar element

(ii) Two dimensional elements: Triangular and rectangular elements are considered as two dimensional elements. These elements are loaded by forces in their own plane. The simplest two dimensional elements have corner nodes as shown in Fig.1.3.

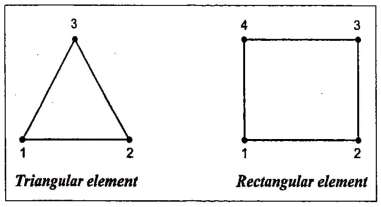


Fig. 1.3. Simple two dimensional elements

(iii) Three dimensional elements: The most common three dimensional elements are tetrahedral and hexahedral (Brick) elements. These elements are used for three dimensional stress analysis problems. The simplest three dimensional elements have corner nodes as shown in Fig.1.4.

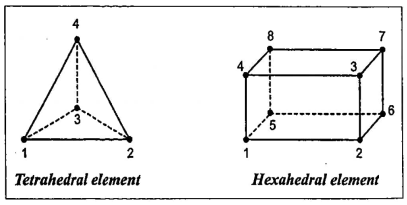


Fig. 1.4. Simple three dimensional elements

(iv) Axisymmetric elements: The axisymmetric element is developed by rotating a triangle or quadrilateral about a fixed axis located in the plane of the element through 360°. It is shown in Fig.1.5. When the geometry and loading of the problems are axisymmetric, these elements are used.

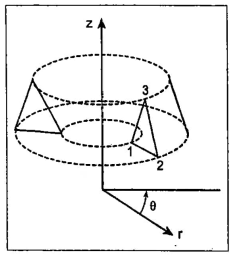


Fig. 1.5. Axisymmetric element

Step 2: Numbering of Nodes and Elements

The nodes and elements should be numbered after discretizaton process. The numbering process is most important since it decide the size of the stiffness matrix and it leads the reduction of memory requirement. While numbering the nodes, the following condition should be satisfied.

$$\left\{\begin{matrix}\text{Maximum}\\ \text{node number}\end{matrix}\right\} - \left\{\begin{matrix}\text{Minimum}\\ \text{node number}\end{matrix}\right\} = \text{Minimum}$$

It is explained in the Fig.1.6(*a*) and (*b*).

Longer Side Numbering Process:

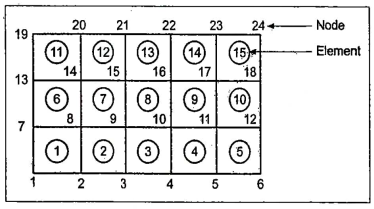
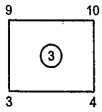


Fig. 1.6. (a)

[*Note:* Number with circle denotes element.
Number without circle denotes node]

Considering element (3),



Maximum node number = 10

Minimum node number = 3

Difference = 7 ... (1.1)

Shorter Side Numbering Process:

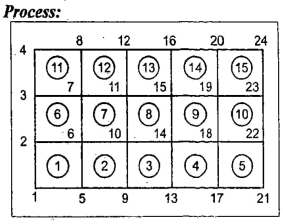
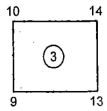


Fig. 1.6. (b)

Considering the same element (3).



Maximum node number = 14

Minimum node number = 9

Difference = 5 ... (1.2)

From equation (1.1) and (1.2), we came to know, shorter side numbering process is followed in the finite element analysis and it reduces the memory requirements.

Step 3: Selection of a Displacement Function or Interpolation Function

✓ It involves choosing a displacement function within each element. Polynomial of linear, quadratic and cubic form are frequently used as displacement functions because they are simple to work within finite element formulation.

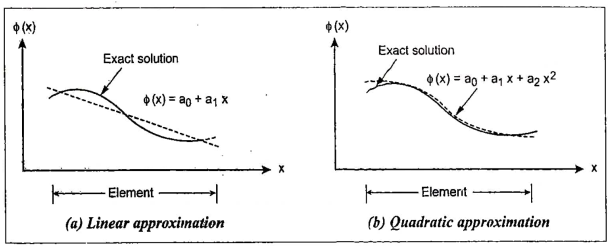


Fig. 1.7. Polynomial approximation in one dimension

✓ The polynomial type of interpolation functions are mostly used due to the following reasons.

1. It is easy to formulate and computerize the finite element equations.
2. It is easy to perform differentiation or integration.

3. The accuracy of the results can be improved by increasing the order of the polynomial.

Fig.1.7 shows the polynomial approximation in one dimension.

Let us consider $\phi(x)$ is a field variable.

Case (i): Linear Polynomial:

One dimensional problem $\phi(x) = a_0 + a_1 x$

Two dimensional problem $\phi(x, y) = a_0 + a_1 x + a_2 y$

Three dimensional problem $\phi(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z$

Case (ii): Quadratic Polynomial:

One dimensional problem $\phi(x) = a_0 + a_1 x + a_2 x^2$

Two dimensional problem $\phi(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy$

Three dimensional problem $\phi(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 yz + a_9 xz$

Step 4: Define the material behaviour by using Strain-Displacement and Stress-Strain Relationships

✓ Strain-Displacement and Stress-Strain relationships are necessary for deriving the equations for each finite element.

✓ In case of one dimensional deformation, the strain-displacement relationship is given by,

$$e = \frac{du}{dx} \qquad \ldots (1.3)$$

where, $u \rightarrow$ Displacement field variable along x direction.

$e \rightarrow$ Strain.

The stress-strain relationship is given by,

$$\sigma = E e \qquad \ldots (1.4)$$

where, $\sigma \rightarrow$ Stress in x direction.

$E \rightarrow$ Modulus of elasticity or Young's modulus.

Step 5: Derivation of element stiffness matrix and equations:

The finite element equation is in matrix form as,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \ldots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \ldots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \ldots & k_{3n} \\ \vdots & & & & \vdots \\ k_{n1} & \ldots & \ldots & \ldots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix}$$

In compact matrix form as,

$$\{ F^e \} = [k^e] \{ u^e \}$$

where, e is a Element, $\{ F \}$ is the vector of element nodal forces, $[k]$ is the element stiffness matrix and $\{ u \}$ is the element displacement vector.

This equation can be derived by any one of the following methods.

(i) Direct Equilibrium Method: This method is much easier to apply for line or one dimensional elements.

(ii) Variational Method: This method is most easily adaptable to the determination of element equations for complicated elements (*i.e.*, element having large number of degrees of freedom) like axisymmetric stress element, plate bending element and two or three dimensional solid stress element.

(iii) Weighted Residual Method: This method is (Galerkin's method) useful for developing the element equations in thermal analysis problems. They are especially useful when a functional such as potential energy is not readily available.

Step 6: Assemble the element equations to obtain the global or total equations:

The individual element equations obtained in step 5 are added together by using a method of superposition *i.e.*, direction stiffness method. The final assembled or global equation which is in the form of

$$\{ F \} = [K] \{ u \} \qquad \ldots (1.5)$$

where, $\{ F \} \rightarrow$ Global force vector.

$[K] \rightarrow$ Global stiffness matrix.

$\{ u \} \rightarrow$ Global displacement vector.

Step 7: Applying boundary conditions:

From equation (1.5), we know that, global stiffness matrix $[K]$ is a singular matrix because its determinant is equal to zero. In order to remove this singularity problem, certain boundary conditions are applied so that the structure remains in place instead of moving as a rigid body. The global equation (1.5) to be modified to account for the boundary conditions of the problem.

Step 8: Solution for the unknown displacements:

A set of simultaneous algebraic equations formed in step 6 can be written in expanded matrix form as follows:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \dots & k_{3n} \\ k_{41} & k_{42} & k_{43} & \dots & k_{4n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ k_{n1} & k_{n2} & k_{n3} & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ \vdots \\ u_n \end{Bmatrix}$$

These equation can be solved and unknown displacements $\{ u \}$ are calculated by using Gaussian elimination method or Gauss-Seidel method.

Step 9: Computation of the element strains and stresses from the nodal displacements, {u}:

In structural stress analysis problem, stress and strain are important factors. From the solution of displacement vector $\{ u \}$, stress and strain value can be calculated.

In case of one dimensional deformation, the strain-displacement relationship is given by,

$$\text{Strain, } e = \frac{du}{dx} \qquad \text{[From equation (1.3)]}$$

$$= \frac{u_2 - u_1}{x_2 - x_1}$$

where, u_1 and u_2 are displacement at node 1 and 2.

$x_2 - x_1$ = Actual length of the element.

From that, we can find the strain value.

By knowing the strain, stress value can be calculated by using the relation,

$$\text{Stress, } \sigma = E\,e$$

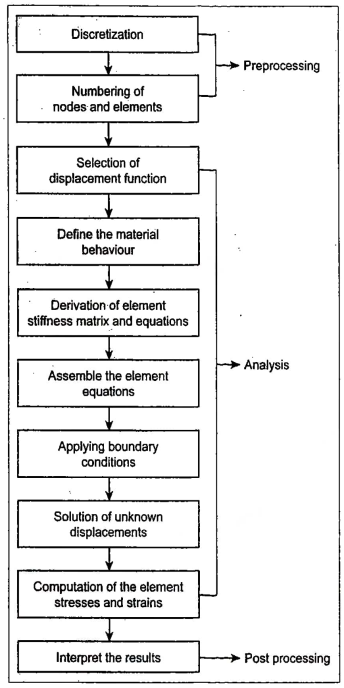
where, E → Young's Modulus.

e → Strain.

Step 10: Interpret the results (Post Processing):

Analysis and evaluation of the solution results is referred to as post-processing. Post processor computer programs help the user to interpret the results by displaying them in graphical form.

Steps 1 to 10 are summarized as follows:



1.4. DISCRETIZATION

1.4.1. Introduction

In this chapter, we are going to learn about discretization, node, assembly, system *etc.* To make this much easier to understand, let us compare these words with the parts over human body. Apart from flesh, our body consists of bones. They are hands, legs, fingers, thigh bones, *etc.* These parts are connected together at different places, so that when movement takes place, we do not feel any pain. Nature has assembled in such a way that every human being is able to sustain certain amount of load without experiencing stain.

Similarly any structure like an automobile, ship, aeroplane, *etc.*, consists of several components assembled together.

Now let us study about 'Element'. The characteristics of an element are as follows:

(*i*) It is a small portion of a system.

(*ii*) It has definite shape.

(*iii*) It should have minimum two nodes.

(*iv*) Nodes are placed where connection is made to another element.

(*v*) Loads act only at the nodes.

Examples:

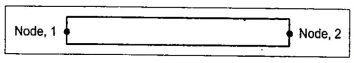
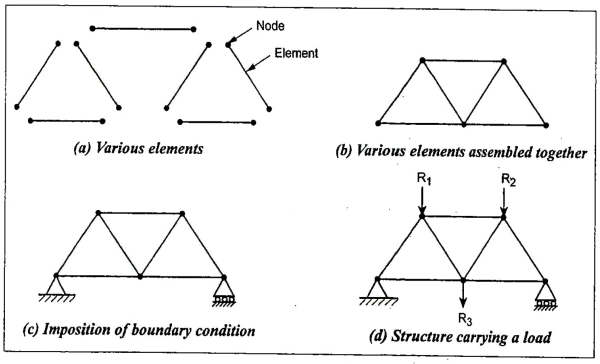


Fig. 1.8. Truss element



(a) Various elements *(b) Various elements assembled together*

(c) Imposition of boundary condition *(d) Structure carrying a load*

Fig. 1.9.

1.4.2. Discretization

The art of subdividing a structure into a convenient number of smaller components is known as Discretization. These smaller components are then put together. The process of uniting the various elements together is called Assemblage. The assemblage of such elements then represents the original body.

Discretization can be classified as follows:

(*i*) Natural.

(*ii*) Artificial (continuum).

1.4.3. Natural Discretization

In structural analysis, a truss is considered as a natural system. The various members of the truss constitute the elements. These elements are connected at various joints known as nodes.

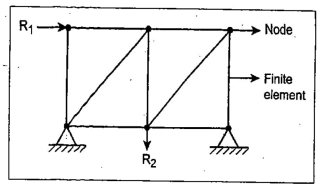


Fig. 1.10. Natural discretization of truss

Nodal Points: Each kind of finite element has a specific structural shape and is interconnected with the adjacent elements by nodal points or nodes.

Nodal forces: The forces that act at each nodal point are called nodal forces.

Degrees of freedom: When the force or reaction act at nodal point, node is subjected to deformation. This deformation includes displacements, rotations, and/or strains. These are collectively known as degrees of freedom or simply we can say nodal displacement is called degrees of freedom.

In Fig.1.10, the truss consists of 9 elements and 6 nodes. There are four freely moving and two extreme constrained nodes. The truss is a natural system as there is no possibility either to increase or decrease the number of elements and the nodes.

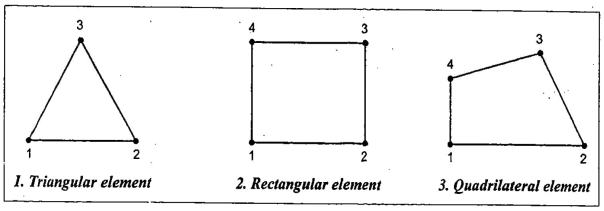
1.4.4. Artificial Discretization (Continuum)

Continuum is generally considered to be a single mass of material as found in a forging, concrete dam, deep beam, plate and so on.

Unlike the truss element which is physically present in the truss, in a continuum, the following three elements exist only in our imagination.

1. Triangular element.
2. Rectangular element.
3. Quadrilateral element.

They are shown in Fig.1.11.



1. Triangular element *2. Rectangular element* *3. Quadrilateral element*

Fig. 1.11.

Discretization using triangular element is shown in Fig.1.12 & 1.13.

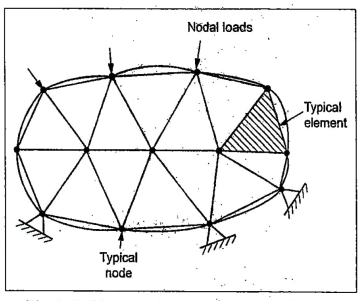
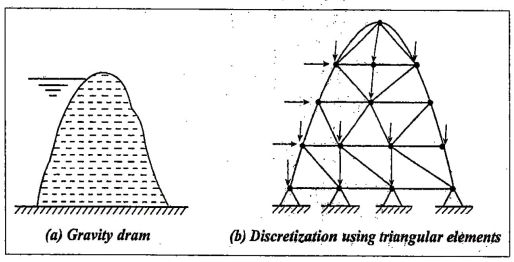


Fig. 1.12. Discretization using triangular elements



(a) Gravity dram *(b) Discretization using triangular elements*

Fig. 1.13.

Fig.1.14 shows a deep beam. In Fig.1.15, it is shown how it is discretized using simple rectangular elements.

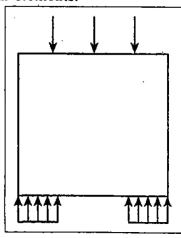


Fig. 1.14. Deep beam

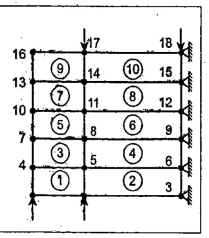
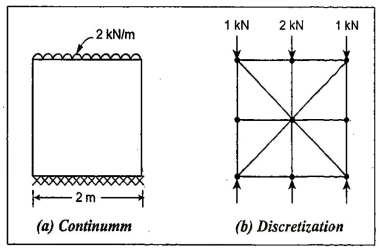


Fig. 1.15. Deep beam discretization using rectangular elements

Fig.1.16 (*a*) shows a planar continuum subjected to uniformly distributed load on the top.

Fig.1.16 (*b*), the continuum is discretized into eight triangular elements. The discretization shown is only one way. We can subdivide the continuum into triangular elements in a number of ways. Alternative way is shown in Fig.1.17.



(a) Continumm *(b) Discretization*

Fig. 1.16.

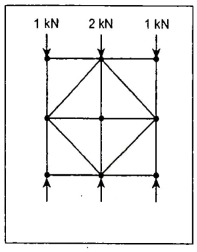


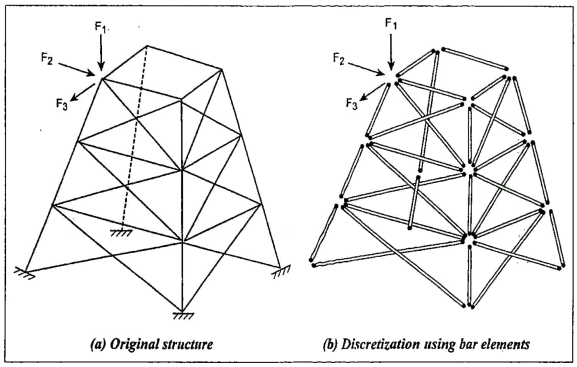
Fig. 1.17. Alternative way of discretization

1.4.5. Discretization Process

The following points to be considered while analysing the discretization process.

(i) Type of elements:

- ✓ The type of elements to be used will be evident from the physical problem.
- ✓ A structure, shown in Fig.1.18 is discretized by using line or bar elements.



(a) Original structure *(b) Discretization using bar elements*

Fig. 1.18.

✓ The finite element idealization can be done by using three dimensional rectangular element in stress analysis of short beam problem which is shown in Fig.1.19.

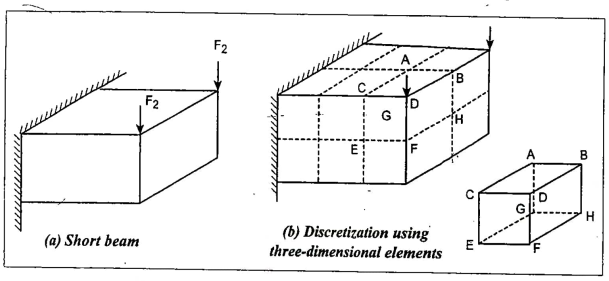


Fig. 1.19. (a) Short beam (b) Discretization using three-dimensional elements

✓ A thin wall sheet shown in Fig.1.20 (*a*), which can be discretized by several types of elements as shown in Fig.1.20(*b*).

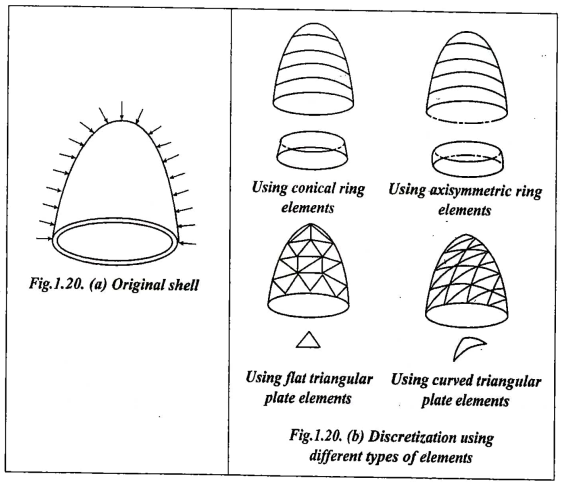


Fig.1.20. (a) Original shell

Fig.1.20. (b) Discretization using different types of elements

✓ The choice of the element to be used for discretization depends upon the following factors.

(*i*) Number of degrees of freedom needed.
(*ii*) Expected accuracy.
(*iii*) Necessary equations required.

✓ However, in certain problems, the given structure cannot be discretized by using only one type of elements. In such cases, we can use two or more types of elements for discretization.

Example: Air craft wing.

(ii) Size of elements:

✓ The size of elements influences the convergence of the solution of the problem directly. So, it should be chosen with more care.

✓ If the size of the element is small, the final solution is more accurate. But the computational time for the smaller size element is more when compared to larger size element.

✓ Another characteristic related to the size of elements that affects the finite element problem solution is the ***"Aspect ratio"*** of the elements.

✓ Aspect ratio is defined as the ratio of the largest dimension of the element to the smallest dimension. The conclusion of many researchers is that the aspect ratio should be close to unity as possible. For a two dimensional rectangular element, the aspect ratio is conveniently defined as length to breadth ratio. Aspect ratio closer to unity yields better results.

(iii) Location of nodes:

✓ If the structure has no abrupt changes in geometric, load, boundary conditions and material properties, the structure can be divided into equal subdivisions. So, the spacing of the nodes are uniform.

✓ If there are any discontinuities in geometric, load, boundary conditions and material properties of the structure, nodes should be introduced at these discontinuities as shown in the following figures.

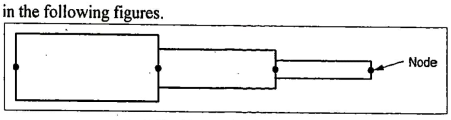


Fig. 1.21. Geometric discontinuities

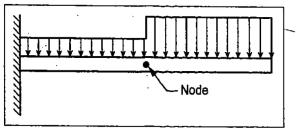


Fig. 1.22. Discontinuity in loading

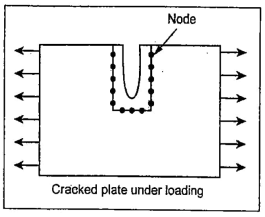


Fig. 1.23. Discontinuity of boundary conditions

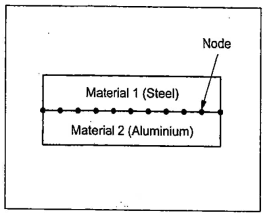


Fig. 1.24. Material discontinuity

(iv) Number of elements:

The number of elements to be selected for discretization depends upon the following factors:

1. Accuracy desired.
2. Size of the elements.
3. Number of degrees of freedom involved.

✓ If the number of element in the structure is increased, the final solution of the problem is expected to be more accurate. But the use of large number of elements involves a large number of degrees of freedom, it leads the storage problem in the available computer memory.

1.5. INITIAL VALUE AND BOUNDARY VALUE PROBLEMS

A differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an initial value problem. The subsidiary conditions are initial conditions. If the subsidiary conditions are given at more than one value of the independent variable, the problem is a boundary value problem and the conditions are boundary conditions.

For Example:

✓ The problem $y'' + 2y' = e^x$; $y(\pi) = 1$, $y'(\pi) = 2$ is an initial-value problem, because both the subsidiary conditions are given at $x = \pi$.

✓ The problem $y'' + 2y' = e^x$; $y(0) = 1$, $y(1) = 1$ is an boundary-value problem, because the two subsidiary conditions are given at the different values $x = 0$ and $x = 1$.

1.6. PROBLEM BASED ON INITIAL VALUE PROBLEM

Example 1.1 ***Find the solution of the initial value problem.***

$$y' + y = 0; \quad y(3) = 2$$

Given: Differential equation, $y' + y = 0$

Boundary condition at $y(3) = 2$

☺ ***Solution:*** Differential equation, $y' + y = 0$ ... (1)

Boundary condition at $y(3) = 2$

$\Rightarrow x = 3, \; y = 2$ ... (2)

Using Auxiliary equation, $\lambda + 1 = 0$ $\quad [\because y' = \frac{d}{dx} = \lambda]$

$$\boxed{\lambda = -1} \quad \text{... (3)}$$

We know that, complementary function or characteristic function,

$$y(x) = c_1 e^{-x} \quad \text{... (4)}$$

Applying the boundary condition (2) in equation (4),

$$y(3) = c_1 e^{-3}$$

$$2 = c_1 e^{-3} \quad [\because y(3) = 2]$$

$$\boxed{c_1 = 2e^3} \quad \text{... (5)}$$

By substituting equation (5) in equation (4)

$$y(x) = 2e^3 e^{-x}$$

As the solution of the initial-value problem.

Result: $y(x) = 2e^3 e^{-x}$, as the solution of the initial-value problem.

Example 1.2 ***Find a solution of the initial-value problem*** $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, ***Boundary conditions*** $y(0) = 2, y'(0) = 5$.

Given: Differential equation, $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

Boundary conditions are $y(0) = 2, \; y'(0) = 5$

☺ ***Solution:*** Differential equation, $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$ ... (1)

Boundary conditions are $y(0) = 2, \; y'(0) = 5$ ... (2)

Using auxiliary equation, $\lambda^2 + \lambda - 2 = 0$

$$(\lambda - 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda_1 = 1, \; \lambda_2 = -2 \quad \text{... (3)}$$

We know that, complementary function, $y(x) = A e^{\lambda_1 x} + B e^{\lambda_2 x}$

Put $\lambda_1 = 1, \lambda_2 = -2$ in the above equation,

$$y(x) = A e^{x} + B e^{-2x} \quad \ldots (4)$$

$$y'(x) = A e^{x} - 2 B e^{-2x} \quad \ldots (5)$$

Applying the boundary conditions (2) in equation (4) and (5), we get

$$y(0) = 2 \Rightarrow x = 0, \; y = 2$$

$$y(0) = A e^{0} + B e^{0}$$

$$2 = A + B$$

$$\boxed{A + B = 2} \quad \ldots (6)$$

Similarly, $y'(0) = 5, \; x = 0, \; y = 5$

$$y'(0) = A e^{0} - 2 B e^{0}$$

$$5 = A - 2B$$

$$\boxed{A - 2B = 5} \quad \ldots (7)$$

Solving the equation (6) and (7),

$$A + B = 2$$

$$A - 2B = 5$$

$$(-) \; (+) \qquad (-)$$

$$3B = -3$$

$$\boxed{B = -1}$$

Substitute the B = –1 value in equation (6),

$$A - 1 = 2$$

$$A = 2 + 1$$

$$\boxed{A = 3}$$

By substituting A and B values in equation (4),

$$\boxed{y(x) = 3 e^{x} - e^{-2x}}$$

Result: General solution $y(x) = 3 e^{x} - e^{-2x}$

1.7. PROBLEMS SOLVED ON BOUNDARY-VALUE PROBLEM

Example 1.3 ***Find a solution of a boundary-value problem $y'' + y = 0$ with $y(0) = 0$ and $y(\pi/6) = 4$.***

Given: Differential equation, $y'' + y = 0$

Boundary conditions are $y(0) = 0, \; y(\pi/6) = 4$

☺ ***Solution:*** Differential equation, $y'' + y = 0 \quad \ldots (1)$

Boundary conditions are $y(0) = 0, \; y(\pi/6) = 4$

Using auxiliary equation, $\lambda^2 + 1 = 0 \quad [\because y' = \frac{d}{dx} = \lambda; \; y'' = \frac{d^2}{dx^2} = \lambda^2]$

$$\lambda^2 = -1$$

$$\lambda = \sqrt{-1}$$

$$\lambda = \pm i$$

$$\boxed{\lambda = \alpha \pm i\beta} \quad [\text{Here, } \alpha = 0, \beta = 1]$$

We know that, complementary functions are,

$$y(x) = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$$

$$y(x) = e^{0} [c_1 \cos x + c_2 \sin x]$$

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x} \quad \ldots (2)$$

Applying boundary conditions in equation (2),

$$y(0) = 0 \Rightarrow y(0) = c_1 \cos(0) + c_2 \sin(0)$$

$$0 = c_1 + 0 \quad [\because \cos 0 = 1, \sin 0 = 0]$$

$$\boxed{c_1 = 0}$$

Similarly, $y(\pi/6) = 4 \Rightarrow y(\pi/6) = c_1 \cos(\pi/6) + c_2 \sin(\pi/6)$

$$4 = c_1 \left(\frac{\sqrt{3}}{2}\right) + c_2 \left(\frac{1}{2}\right)$$

Put $c_1 = 0, \quad c_2 \left(\frac{1}{2}\right) = 4$

$$\boxed{c_2 = 8}$$

Substitute the c_1 and c_2 values in equation (2)

$$y(x) = 0 + 8 \sin x$$

$$\boxed{y(x) = 8 \sin x}$$

Result: General equation, $y(x) = 8 \sin x$

Example 1.4 ***Find the solution of the boundary-value problem $y'' + 4y = 0$ with $y(\pi/8) = 0, y(\pi/6) = 1$.***

Given: Differential equation, $y'' + 4y = 0 \quad \ldots (1)$

Boundary conditions are, $y(\pi/8) = 0$ and $y(\pi/6) = 1 \quad \ldots (2)$

☺ ***Solution:*** Differential equation, $y'' + 4y = 0$

Boundary conditions are, $y(\pi/8) = 0$ and $y(\pi/6) = 1$

Using auxiliary equation, $\lambda^2 + 4 = 0$

$$\lambda^2 = -4$$

$$\lambda = \sqrt{-4}$$

$$\lambda = \pm 2i$$

$$\boxed{\lambda = \alpha \pm i\beta} \qquad [\because \text{ Here, } \alpha = 0, \beta = 2]$$

We know that, complementary functions,

$$y(x) = e^{\alpha x}\,[c_1 \cos \beta x + c_2 \sin \beta x]$$

$$y(x) = e^{0}\,[c_1 \cos 2x + c_2 \sin 2x]$$

$$\boxed{y(x) = c_1 \cos 2x + c_2 \sin 2x} \qquad \ldots (3)$$

Applying first boundary condition in equation (3),

$$y(\pi/8) = 0$$

$$\Rightarrow y(\pi/8) = c_1 \cos 2(\pi/8) + c_2 \cos 2(\pi/8)$$

$$y(\pi/8) = c_1 \cos(\pi/4) + c_2 \sin(\pi/4)$$

$$0 = c_1\left(\frac{1}{2} \times \sqrt{2}\right) + c_2\left(\frac{1}{2} \times \sqrt{2}\right)$$

$$\boxed{c_1\left(\frac{\sqrt{2}}{2}\right) + c_2\left(\frac{\sqrt{2}}{2}\right) = 0}$$

$$\text{(or)} \quad c_1\left(\frac{1}{\sqrt{2}}\right) + c_2\left(\frac{1}{\sqrt{2}}\right) = 0 \qquad \ldots (4)$$

Applying second boundary condition in equation (3),

$$y(\pi/6) = 1$$

$$\Rightarrow y(\pi/6) = c_1 \cos 2(\pi/6) + c_2 \sin 2(\pi/6)$$

$$1 = c_1 \cos\frac{\pi}{3} + c_2 \sin\frac{\pi}{3}$$

$$c_1\left(\frac{\sqrt{3}}{2}\right) + c_2\left(\frac{1}{2}\right) = 1 \qquad \ldots (5)$$

Solving the equation (4) and (5),

Equation (4) $\Rightarrow$ $c_1\left(\frac{1}{\sqrt{2}}\right) + c_2\left(\frac{1}{\sqrt{2}}\right) = 0$

Equation (5) $\Rightarrow$ $c_1\left(\frac{\sqrt{3}}{2}\right) + c_2\left(\frac{1}{2}\right) = 1$

Get, $\boxed{c_1 = 2.732}$

Substitute the c_1 value in equation (5)

$$2.732\left(\frac{\sqrt{3}}{2}\right) + c_2\left(\frac{1}{2}\right) = 1$$

$$\boxed{c_2 = -2.732}$$

From these c_1 and c_2 value, substitute the equation (3)

$$y(x) = 2.732 \cos 2x + (-2.732) \sin 2x$$

$$y(x) = 2.732\,[\cos 2x - \sin 2x]$$

Result: General equation, $y(x) = 2.732\,[\cos 2x - \sin 2x]$

1.8. EIGEN VALUE PROBLEM [BOUNDARY VALUE PROBLEM]

When applied to the boundary-value problem, has the form

$$y'' + P(x, \lambda)\,y' + Q(x, \lambda)\,y = 0$$

Non-trivial solutions may exit for certain values of λ but not for other values of λ. Those values of λ for which non-trivial solutions do exists are called eigen values, the corresponding non-trivial solutions are called eigen functions.

For example: For the axial vibration of a bar, to find $u(x)$ and λ that satisfies the partial differentiation equation and boundary conditions are,

$$A\,y'' + \lambda\,y' = 0, \quad \text{for } 0 < x < L$$

$$y(0) = 0, \quad y'(L) = 0$$

1.9. PROBLEMS SOLVED ON EIGEN VALUE PROBLEM [BOUNDARY VALUE PROBLEM]

Example 1.5 ***Find the eigen values and eigen function of $y'' - 4xy' + 4\lambda^2 y = 0$; with boundary conditions are, $y(0) = 0$, $y(1) + y'(1) = 0$.***

Given: Differential equation,

$$y'' - 4xy' + 4\lambda^2 y = 0$$

Boundary conditions are, $y(0) = 0$

$$y(1) + y'(1) = 0$$

☺ ***Solution:*** Given differential equation,

$$y'' - 4xy' + 4\lambda^2 y = 0 \qquad \ldots (1)$$

Boundary conditions are, $y(0) = 0$

$$y(1) + y'(1) = 0 \qquad \ldots (2)$$

The auxiliary equation is,

$$m^2 - 4\lambda m + 4\lambda^2 = 0 \qquad \ldots (3)$$

$$\Rightarrow (m - 2\lambda)(m - 2\lambda) = 0$$

$$\Rightarrow (m - 2\lambda) = 0 \text{ and } (m - 2\lambda) = 0$$

$$\Rightarrow m_1 = 2\lambda,\ m_2 = 2\lambda$$

We know that, complementary function is,

$$y(x) = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$$

$$y(x) = c_1 e^{2\lambda x} + c_2 x e^{2\lambda x} \quad \ldots (4)$$

Differentiate with respect to "x" in equation (4),

$$y'(x) = 2\lambda c_1 e^{2\lambda x} + c_2 [x 2\lambda e^{2\lambda x} + e^{2\lambda x}]$$

$$y'(x) = 2\lambda c_1 e^{2\lambda x} + c_2 [x 2\lambda e^{2\lambda x} + e^{2\lambda x}] \quad \ldots (5)$$

Applying first boundary condition in equation (5),

$$y(0) = 0, \; x = 0, \; y = 0$$

$$\Rightarrow y(0) = c_1 e^0 + c_2(0) e^0$$

$$\boxed{c_1 = 0}$$

Applying second boundary conditions in equation (5),

$$y(1) + y'(1) = 0$$

We get,

$$y(1) = c_1 e^{2\lambda} + c_2 e^{2\lambda}$$

$$y'(1) = 2\lambda c_1 e^{2\lambda} + c_2 (2\lambda e^{2\lambda} + e^{2\lambda})$$

We know that,

$$\Rightarrow \quad c_1 e^{2\lambda} + c_2 e^{2\lambda} + 2\lambda c_1 e^{2\lambda} + c_2 (2\lambda e^{2\lambda} + e^{2\lambda}) = 0$$

$$\Rightarrow \quad c_1 (1 + 2\lambda) + c_2 (2 + 2\lambda) = 0$$

If now follows that $c_1 = 0$ and either $c_2 = 0$ (or) $\lambda = -1$.

The choice of $c_2 = 0$.

The result in the trivial solution $y = 0$.

The choice of $\lambda = -1$.

The result in the non-trivial solution, *i.e.*, $y = c_2 x e^{-2x}$, where $c_2 \Rightarrow$ arbitrary

Thus the boundary value problem has eigen value $\lambda = -1$ and the eigen function

$$\boxed{y = c_2 x e^{-2x}}$$

Result: Eigen value and eigen functions, $y = c_2 x e^{-2x}$

Example 1.6 ***Find the eigen value and eigen function of $y'' - 4\lambda y' + 4\lambda^2 y = 0$; with boundary conditions are, $y'(1) = 0$, $y(2) + 2y'(2) = 0$.***

Given: Differential equation, $y'' - 4\lambda y' + 4\lambda^2 y = 0$

Boundary conditions are, $y'(1) = 0, \; y(2) + 2y'(2) = 0$

☺ ***Solution:*** Differential equation,

$$y'' - 4\lambda y' + 4\lambda^2 y = 0 \quad \ldots (1)$$

Boundary conditions are, $y'(1) = 0$

$$y(2) + 2y'(2) = 0 \quad \ldots (2)$$

The auxiliary equation, $m^2 - 4\lambda m + 4\lambda^2 = 0 \quad \ldots (3)$

$$(m - 2\lambda)(m - 2\lambda) = 0$$

$$m_1 = 2\lambda, \; m_2 = 2\lambda$$

We know that, complementary functions are,

$$y(x) = c_1 e^{2\lambda x} + c_2 x e^{2\lambda x} \quad \ldots (4)$$

Differentiate with respect to 'x' in equation (4),

$$y'(x) = 2\lambda c_1 e^{2\lambda x} + c_2 [x 2\lambda e^{2\lambda x} + e^{2\lambda x}] \quad \ldots (5)$$

Applying first boundary condition in equation (5), we get,

$$y'(1) = 0$$

$$\Rightarrow \quad y'(1) = 2\lambda c_1 e^{2\lambda} + c_2 (2\lambda e^{2\lambda} + e^{2\lambda}) = 0$$

$$\Rightarrow \quad 2\lambda c_1 e^{2\lambda} + c_2 [2\lambda e^{2\lambda} + e^{2\lambda}] = 0$$

$$\Rightarrow \quad 2\lambda c_1 + c_2 [2\lambda + 1] = 0 \quad \ldots (6)$$

Applying second boundary condition in equation (5) and (4),

$$y(2) + 2y'(2) = 0$$

$$\Rightarrow \quad y(2) = c_1 e^{4\lambda} + 2c_2 e^{4\lambda}$$

$$\Rightarrow \quad y'(2) = 2\lambda c_1 e^{4\lambda} + c_2 [4\lambda e^{4\lambda} + e^{4\lambda}]$$

Adding both equations,

$$\Rightarrow \quad c_1 e^{4\lambda} + 2c_2 e^{4\lambda} + 2[\lambda c_1 e^{4\lambda} + c_2 (4\lambda e^{4\lambda} + e^{4\lambda})] = 0$$

$$\Rightarrow \quad c_1 e^{4\lambda} + 2c_2 e^{4\lambda} + 4\lambda c_1 e^{4\lambda} + 2c_2 (4\lambda e^{4\lambda} + e^{4\lambda}) = 0$$

$$\Rightarrow \quad c_1 e^{4\lambda} + 2c_2 e^{4\lambda} + 4\lambda c_1 e^{4\lambda} + 8\lambda c_2 e^{4\lambda} + 2c_2 e^{4\lambda} = 0$$

$$\Rightarrow \quad c_1 + 2c_2 + 4\lambda c_1 + 8\lambda c_2 + 2c_2 = 0$$

$$\Rightarrow \quad (1 + 4\lambda) c_1 + (8\lambda + 4) c_2 = 0 \quad \ldots (7)$$

Solving equations (6) and (7),

$$\Rightarrow \quad 2\lambda c_1 + c_2 (2\lambda + 1) = 0$$

$$\Rightarrow \quad (1 + 4\lambda) c_1 + (8\lambda + 4) c_2 = 0$$

Set determination is equal to zero,

$$\begin{vmatrix} 2\lambda & 1 + 2\lambda \\ 1 + 4\lambda & 4 + 8\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 2\lambda (4 + 8\lambda) - (1 + 2\lambda)(1 + 4\lambda) = 0$$

$$\Rightarrow \quad 8\lambda (1 + 2\lambda) - (1 + 2\lambda)(1 + 4\lambda) = 0$$

$$\Rightarrow \quad (1 + 2\lambda)[8\lambda - (1 + 4\lambda)] = 0$$

$$\Rightarrow \quad (1+2\lambda)(4\lambda-1) = 0$$

$$\Rightarrow \quad (1+2\lambda) = 0 \text{ and } (4\lambda-1)=0$$

$$\boxed{\lambda_1 = -\frac{1}{2} \text{ and } \lambda_2 = \frac{1}{4}}$$

When, $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{1}{4}$ the result has non-trivial solution.

It follows that eigen values are $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{1}{4}$ and the corresponding eigen functions are,

$$y_1 = c_2 x e^{-x} \quad \text{and}$$

$$y_2 = c_2(-3+x)e^{x/2}$$

Result: Eigen values, $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = \frac{1}{4}$

Eigen functions, $y_1 = c_2 x e^{-x}$

$$y_2 = c_2(-3+x)e^{x/2}$$

1.10. SOLUTION OF EIGEN VALUE PROBLEMS [MATRIX APPROACH]

There are three methods to solve eigen value problems. They are,

1. Determinant methods
2. Transformation methods
3. Vector iteration methods

1.10.1. Determinant Methods

These methods are primarily based on the equation,

$$\{[K]-\lambda[m]\}\{u\} = 0$$

If the eigen vector is non-trivial, the required condition is,

$$\det|[K]-\lambda[m]| = 0$$

$$\Rightarrow |[K]-\lambda[m]| = 0 \qquad \ldots (1.6)$$

Trial value of λ is taken and the determinant $|[K]-\lambda[m]| = 0$ is computed. The curve is drawn by taking several trial values.

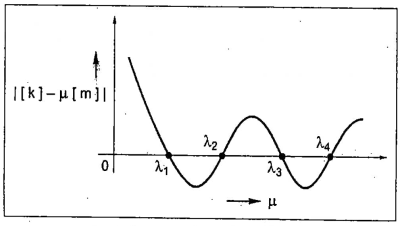


Fig. 1.25. Determinant-based Method

Due to heavy computational cost and several iterations are required to determine all the eigen values, the determinant based methods are not implemented in practice.

1.10.2. Transformation Methods

This method is used to transform the eigen value problems,

$$\text{Let,} \quad [K]\{u\} = \lambda\{u\} \qquad \ldots (1.7)$$

Transform [K] into a diagonal matrix by using a series of matrix transformations,

$$[K] = [T]^T[K][T] \qquad \ldots (1.8)$$

where, [T] is the transformation matrix, which is usually an orthogonal matrix.

$$i.e., \quad [T]^T = [T]^{-1}$$

When we transform [K] completely into diagonal matrix, then the elements on the diagonals are considered as eigen values,

$$[T]^T[K][T] = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} \qquad \ldots (1.9)$$

where, λ_1, λ_2 and λ_3 are eigen values.

1.10.3. Vector Iteration Methods

- ✓ Vector iteration methods are normally available in many commercial finite element software packages.
- ✓ In this method, trial eigen vector is assumed and repeated matrix manipulation is performed to compute the desired eigen vector.

1.11. SOLVED PROBLEMS ON EIGEN VALUES [MATRIX APPROACH]

Example 1.7 ***Find the eigen values of*** $\begin{pmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{pmatrix}$.

☺***Solution:***

Step 1: To find characteristic equation,

Let the given matrix be $A = \begin{pmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{pmatrix}$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \qquad \ldots (1)$$

where, a_1 = Sum of leading diagonal elements

$$= 4 + 10 - 13$$

$$\boxed{a_1 = 1} \quad \ldots (2)$$

a_2 = Sum of minors of the leading diagonal elements

$$= \begin{vmatrix} 10 & 4 \\ -30 & -13 \end{vmatrix} + \begin{vmatrix} 4 & -10 \\ 6 & -13 \end{vmatrix} + \begin{vmatrix} 4 & -20 \\ -2 & 10 \end{vmatrix}$$

$$= -130 + 120 - 52 + 60 + 40 - 40$$

$$\boxed{a_2 = -2} \quad \ldots (3)$$

$$a_3 = |A| = \begin{vmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{vmatrix}$$

$$= 4[-130 + 120] + 20[26 - 24] - 10[60 - 60]$$

$$= -40 + 40 + 0$$

$$\boxed{a_3 = 0} \quad \ldots (4)$$

Substitute the a_1, a_2 and a_3 values in equation (1),

$$\boxed{\lambda^3 - \lambda^2 - 2\lambda = 0}$$

Step 2: To find eigen values:

$$\lambda^3 - \lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda^2 - \lambda - 2) = 0$$

When $\lambda = 0$, $\quad \lambda^2 - \lambda - 2 = 0$

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$$= 2 \text{ or } -1$$

$\therefore$ Eigen values are $\boxed{\lambda = 0, -1, 2}$

Result: Eigen values, $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 2$.

1.12. WEIGHTED RESUDUAL METHODS

1.12.1. Introduction

The method of weighted residuals is a powerful approximate procedure applicable to several problems. For structural problems, potential energy functional can be easily formed, so, Rayleigh-Ritz method is used. On the other hand for non-structural problems, the differential equation of the phenomenon can be easily formulated. For such type of problems, the method of weighted residuals becomes very useful. There are many types of weighted residuals, of them four are very popular. They are:

(*i*) Point collocation method.

(*ii*) Subdomain collocation method.

(*iii*) Least squares method.

(*iv*) Galerkin's method.

Among these four methods, the Galerkin approach has the widest choice and is used in finite element analysis.

1.12.2. General Procedure

Our interest is to find y, which is the solution for the differential equation. If it is not possible to find a solution, we assume an approximate function for y. When we substitute the approximate solution in the differential equation, we can get residual and that residual can be expressed as,

$$R(x_i;\ a_1,\ a_2,\ a_3) = 0$$

where a_1, a_2 are unknown parameters present in assumed trial function.

The assumed trial function can be expressed as follows:

$$y = f(x; a_1, a_2, a_3, \ldots\ldots, a_n)$$

Trial function y must exactly satisfy the boundary conditions.

The method of weighted residuals needs the parameters a_1, a_2, a_3, ..., a_n, to be determined by satisfying the following equation.

$$\int_D w_i R(x; a_1, a_2, a_3, \ldots\ldots, a_n)\, dx = 0 \quad \ldots (1.10)$$

where, w_i is a function of x and known as weighting function.

D is a domain; R is a residual.

1.12.3. Point Collocation Method

In the collocation method, also called point collocation, residuals are set to zero at n different locations X_i, and the weighting function w_i is denoted as $\delta(x - x_i)$.

$$\Rightarrow w_i = \delta(x - x_i)$$

Substituting w_i value in equation (1.10),

$$\Rightarrow \int_D \delta(x - x_i) R(x;\ a_1, a_2, a_3, \ldots\ldots, a_n)\, dx = 0 \quad \ldots (1.11)$$

The x_i's are referred to as collocation points and are selected by the discretion of the analyst.

In equation (1.11), term $\int_D \delta(x - x_i) = 1$

So, $R(x; a_1, a_2, a_3, \ldots\ldots, a_n) = 0$

1.12.4. Subdomain Collocation Method

In this method, the weighting functions (w_i) are chosen to be unity over a portion of the domain and zero elsewhere. It is given as follows:

$$w_1 = \begin{cases} 1 & \text{for } x \text{ in } D_1 \\ 0 & \text{for } x \text{ not in } D_1 \end{cases}$$

$$w_2 = \begin{cases} 1 & \text{for } x \text{ in } D_2 \\ 0 & \text{for } x \text{ not in } D_2 \end{cases}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$w_n = \begin{cases} 1 & \text{for } x \text{ in } D_n \\ 0 & \text{for } x \text{ not in } D_n \end{cases}$$

where D is a domain.

1.12.5. Least Squares Method

In this method, the integral of the weighted square of the residual over the domain is required to be minimum.

$$i.e., \quad I = \int_D [R(x; a_1, a_2, a_3, \ldots\ldots a_n)]^2 \, dx = \text{minimum}$$

$$\text{where,} \quad I = f(a_1, a_2, \ldots\ldots a_n)$$

$$\text{The requirement is } \frac{\partial I}{\partial a_i} = 0, \quad i = 1, 2, 3, \ldots\ldots n$$

1.12.6. Galerkin's Method

In this method, the trial function, $N_i(x)$, itself is considered as the weighting functions; that is,

$$w_i = N_i(x)$$

Substitute w_i value in equation (1.10),

$$\Rightarrow \int_D N_i(x) \, R(x; a_1, a_2, \ldots\ldots a_n) \, dx = 0 \qquad \ldots (1.12)$$

$$i = 1, 2, 3, \ldots\ldots n$$

1.12.7. SOLVED PROBLEMS - WEIGHTED RESIDUAL METHOD

Example 1.8 ***The following differential equation is available for a physical phenomenon*** $A E \frac{d^2y}{dx^2} + q_0 = 0$ ***with the boundary conditions***

$$y(0) = 0$$

$$\left.\frac{dy}{dx}\right|_{x=L} = 0$$

Find the value of f(x) using the weighted residual method.

Given: Differential equation, $A E \frac{d^2y}{dx^2} + q_0 = 0$

Boundary conditions are $y(0) = 0$

$$\left.\frac{dy}{dx}\right|_{x=L} = 0$$

To find: $f(x)$

☺***Solution:*** Assume a trial solution.

$$\text{Let} \quad y(x) = a_0 + a_1 x + a_2 x^2 \qquad \ldots (1)$$

Apply first boundary condition, *i.e.*, substitute $x = 0$ and $y = 0$ in equation (1).

$$(1) \Rightarrow \quad 0 = a_0 + 0 + 0$$

$$\Rightarrow \quad \boxed{a_0 = 0}$$

Apply second boundary condition,

$$y(x) = a_0 + a_1 x + a_2 x^2$$

$$\frac{dy}{dx} = a_1 + 2 a_2 x$$

$$\text{At } x = L, \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 0 = a_1 + 2 a_2 L$$

$$\Rightarrow \quad \boxed{a_1 = -2 a_2 L}$$

Substitute a_0 and a_1 value in equation (1),

$$(1) \Rightarrow \quad y(x) = -2 a_2 x L + a_2 x^2$$

$$\boxed{y(x) = a_2 [x^2 - 2 x L]} \qquad \ldots (2)$$

$$\Rightarrow \quad \frac{dy}{dx} = a_2 (2x - 2L)$$

$$\boxed{\frac{d^2y}{dx^2} = 2 a_2}$$

4.

We know that, Residual, $R = A E \frac{d^2y}{dx^2} + q_0 = 0$

$$\Rightarrow A E (2 a_2) + q_0 = 0$$

$$\Rightarrow A E 2 a_2 = -q_0$$

$$\Rightarrow \boxed{a_2 = \frac{-q_0}{2 A E}}$$

Substitute a_2 value in equation (2),

$$\Rightarrow y(x) = \frac{-q_0}{2 A E} [x^2 - 2 x L]$$

$$\boxed{y(x) = \frac{q_0}{2 A E} [2 x L - x^2]}$$

Result: Final solution, $y(x) = \frac{q_0}{2 A E} [2 x L - x^2]$

Example 1.9 ***The governing differential equation for the fully developed laminar flow is given by*** $\mu \frac{d^2u}{dx^2} + \rho g \cos\theta = 0.$

If boundary conditions are $\left.\frac{du}{dx}\right|_{x=0} = 0$, $u(L) = 0$, ***find the velocity distribution, u(x).***

☺***Solution:*** Differential equation

$$\mu \frac{d^2u}{dx^2} + \rho g \cos\theta = 0$$

Boundary conditions $\left.\frac{du}{dx}\right|_{x=0} = 0$

$$u(L) = 0$$

To find: Velocity distribution, $u(x)$.

☺ ***Solution:*** Assume a trial function.

$$\text{Let } u(x) = a_0 + a_1 x + a_2 x^2 \ldots\ldots \quad \ldots (1)$$

Apply first boundary condition,

$$i.e., \frac{du}{dx} = 0 \text{ at } x = 0.$$

$$\Rightarrow \frac{du}{dx} = a_1 + 2 a_2 x$$

$$\text{At } x = 0, \frac{du}{dx} = 0$$

$$\Rightarrow 0 = a_1$$

$$\Rightarrow \boxed{a_1 = 0}$$

Apply secondary boundary condition

$$i.e., \text{ at } x = L \quad u(x) = 0$$

$$\Rightarrow u(x) = a_0 + a_1 x + a_2 x^2$$

$$\Rightarrow 0 = a_0 + a_1 L + a_2 L^2$$

Substitute $a_1 = 0$

$$\Rightarrow 0 = a_0 + a_2 L^2$$

$$\Rightarrow \boxed{a_0 = -a_2 L^2}$$

Substitute a_0 and a_1 value in equation (1),

$$u(x) = -a_2 L^2 + 0 + a_2 x^2$$

$$= a_2 [x^2 - L^2]$$

$$\boxed{u(x) = a_2 [x^2 - L^2]} \quad \ldots (2)$$

$$\Rightarrow \frac{du}{dx} = a_2 [2 x]$$

$$\boxed{\frac{d^2u}{dx^2} = 2 a_2}$$

We know that, Residual, $R = \mu \frac{d^2u}{dx^2} + \rho g \cos\theta = 0$

$$\Rightarrow \mu (2 a_2) + \rho g \cos\theta = 0$$

$$\mu 2 a_2 = -\rho g \cos\theta$$

$$\boxed{a_2 = \frac{-\rho g \cos\theta}{2 \mu}}$$

Substitute a_2 value in equation (2),

$$\Rightarrow u(x) = \frac{-\rho g \cos\theta}{2 \mu} [x^2 - L^2]$$

$$u(x) = \frac{\rho g \cos\theta}{2 \mu} [L^2 - x^2]$$

Result: Velocity distribution $u(x) = \frac{\rho g \cos\theta}{2 \mu} [L^2 - x^2]$

Example 1.10 ***Find the solution for the following differential equation.***

$$E I \frac{d^4u}{dx^4} - q_0 = 0$$

The boundary conditions are $u(0) = 0, \quad \frac{du}{dx}(0) = 0$

$$\frac{d^2u}{dx^2}(L) = 0; \quad \frac{d^3u}{dx^3}(L) = 0$$

Given: The governing differential equation

$$E\,I\,\frac{d^4u}{dx^4} - q_0 = 0$$

The boundary conditions are $u(0) = 0;\quad \frac{du}{dx}(0) = 0$

$$\frac{d^2u}{dx^2}(L) = 0;\quad \frac{d^3u}{dx^3}(L) = 0$$

☺***Solution:*** Assume a trial function.

$$\text{Let}\quad u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots \qquad \ldots (1)$$

Apply first boundary condition,

$$i.e.,\ \text{at } x = 0,\ u(x) = 0$$

$$\Rightarrow\quad 0 = a_0 + 0 + 0 + 0 + 0$$

$$\Rightarrow\quad \boxed{a_0 = 0}$$

Apply second boundary condition, *i.e.*, at $x = 0$, $\frac{du}{dx} = 0$.

$$\Rightarrow\quad \frac{du}{dx} = 0 + a_1 + 2\,a_2 x + 3\,a_3 x^2 + 4\,a_4 x^3$$

$$0 = a_1 + 0 + 0 + 0$$

$$\Rightarrow\quad \boxed{a_1 = 0}$$

Apply third boundary condition *i.e.*, at $x = L$, $\frac{d^2u}{dx^2} = 0$.

$$\Rightarrow\quad \frac{d^2u}{dx^2} = 2\,a_2 + 6\,a_3 x + 12\,a_4 x^2$$

$$\Rightarrow\quad 0 = 2\,a_2 + 6\,a_3 L + 12\,a_4 L^2$$

$$\Rightarrow\quad 2\,a_2 = -6\,a_3 L - 12\,a_4 L^2$$

$$\Rightarrow\quad \boxed{a_2 = -[3\,a_3 L + 6\,a_4 L^2]}$$

Apply forth boundary condition, *i.e.*, at $x = L$, $\frac{d^3u}{dx^3} = 0$.

$$\Rightarrow\quad \frac{d^3u}{dx^3} = 0 + 6\,a_3 + 24\,a_4 x$$

$$\Rightarrow\quad 0 = 6\,a_3 + 24\,a_4 L$$

$$\Rightarrow\quad 6\,a_3 = -24\,a_4 L$$

$$\Rightarrow\quad \boxed{a_3 = -4\,a_4 L}$$

Substitute a_0, a_1, a_2 and a_3 values in equation (1),

$$u(x) = 0 + 0 - [\,3\,a_3 L + 6\,a_4 L^2\,]\,x^2 - 4\,a_4 L\,x^3 + a_4 x^4$$

$$= -[\,3\,a_3 L + 6\,a_4 L^2\,]\,x^2 - 4\,a_4 L\,x^3 + a_4 x^4$$

$$= -[\,3\,(-4\,a_4 L) \times L + 6\,a_4 L^2\,]\,x^2 - 4\,a_4 L\,x^3 + a_4 x^4$$

$$[\because a_3 = -4\,a_4 L]$$

$$= 12\,a_4 L^2 x^2 - 6\,a_4 L^2 x^2 - 4\,a_4 L\,x^3 + a_4 x^4$$

$$= a_4\,[\,12\,L^2 x^2 - 6\,L^2 x^2 - 4\,L\,x^3 + x^4\,]$$

$$= a_4\,[\,6\,L^2 x^2 - 4\,L\,x^3 + x^4\,]$$

$$\boxed{u(x) = a_4\,[\,6\,L^2 x^2 - 4\,L\,x^3 + x^4\,]} \qquad \ldots (2)$$

$$\Rightarrow\quad \frac{du}{dx} = a_4\,[\,6\,L^2\,(2\,x) - 12\,L\,x^2 + 4\,x^3\,]$$

$$\Rightarrow\quad \frac{d^2u}{dx^2} = a_4\,[\,6\,L^2\,(2) - 24\,L\,x + 12\,x^2\,]$$

$$\Rightarrow\quad \frac{d^3u}{dx^3} = a_4\,[\,0 - 24\,L + 24\,x\,]$$

$$\Rightarrow\quad \frac{d^4u}{dx^4} = a_4\,[\,0 - 0 + 24\,]$$

$$\boxed{\frac{d^4u}{dx^4} = 24\,a_4}$$

We know that, Residual, $R = E\,I\,\frac{d^4u}{dx^4} - q_0 = 0$

$$\Rightarrow\quad E\,I\,(24\,a_4) - q_0 = 0$$

$$\Rightarrow\quad E\,I\,24\,a_4 = q_0$$

$$\Rightarrow\quad \boxed{a_4 = \frac{q_0}{24\,E\,I}}$$

Substitute a_4 value in equation (2),

$$\Rightarrow\quad u(x) = \frac{q_0}{24\,E\,I}\,[\,6\,L^2 x^2 - 4\,L\,x^3 + x^4\,]$$

$$\Rightarrow\quad \boxed{u(x) = \frac{q_0}{24\,E\,I}\,[\,x^4 - 4\,L\,x^3 + 6\,L^2 x^2\,]}$$

Result: Final solution $u(x) = \frac{q_0}{24\,E\,I}\,[\,x^4 - 4\,L\,x^3 + 6\,L^2 x^2\,]$

Example 1.11 ***The following differential equation is available for a physical phenomenon***

$$A\,E\frac{d^2u}{dx^2} + a\,x = 0$$

The boundary conditions are $u(0) = 0,\ A\,E\left.\frac{du}{dx}\right|_{x=L} = 0.$

By using Galerkin's technique, find the solution of the above differential equation.

Given: Differential equation, $A\,E\frac{d^2u}{dx^2} + a\,x = 0$

Boundary conditions, $u(0) = 0,\ A\,E\left.\frac{du}{dx}\right|_{x=L} = 0$

To find: $u(x)$ by using Galerkin's technique.

☺***Solution:*** Assume a trial function.

$$\text{Let}\quad u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \qquad \ldots (1)$$

Apply first boundary condition, *i.e.*, at $x = 0$, $u(x) = 0$.

$$(1) \Rightarrow \quad 0 = a_0 + 0 + 0 + 0$$

$$\Rightarrow \quad \boxed{a_0 = 0}$$

Apply second boundary condition, *i.e.*, at $x = L$, $A\,E\frac{du}{dx} = 0$.

$$\Rightarrow \quad \frac{du}{dx} = 0 + a_1 + 2\,a_2 x + 3\,a_3 x^2$$

$$\Rightarrow \quad 0 = a_1 + 2\,a_2 L + 3\,a_3 L^2$$

$$\Rightarrow \quad \boxed{a_1 = -(2\,a_2 L + 3\,a_3 L^2)}$$

Substitute a_0 and a_1 value in equation (1),

$$u(x) = 0 + -(2\,a_2 L + 3\,a_3 L^2)\,x + a_2 x^2 + a_3 x^3$$

$$= -2\,a_2 L x - 3\,a_3 L^2 x + a_2 x^2 + a_3 x^3$$

$$= a_2\,[x^2 - 2\,L\,x] + a_3\,[x^3 - 3\,L^2 x]$$

$$\boxed{u(x) = a_2\,[x^2 - 2\,L\,x] + a_3\,[x^3 - 3\,L^2 x]} \qquad \ldots (2)$$

We know that, $\quad \text{Residual, } R = A\,E\frac{d^2u}{dx^2} + a\,x \qquad \ldots (3)$

$$(2) \Rightarrow \quad \frac{du}{dx} = a_2\,[2\,x - 2\,L] + a_3\,[3\,x^2 - 3\,L^2]$$

$$\frac{d^2u}{dx^2} = a_2\,[2] + a_3\,[6\,x]$$

$$\boxed{\frac{d^2u}{dx^2} = 2\,a_2 + 6\,a_3 x}$$

Substitute $\frac{d^2u}{dx^2}$ value in equation (3),

$$(3) \Rightarrow \quad R = A\,E\,(2\,a_2 + 6\,a_3 x) + a\,x$$

$$\text{Residual, } R = A\,E\,(2\,a_2 + 6\,a_3 x) + a\,x \qquad \ldots (4)$$

From Galerkin's technique,

$$\int_0^L w_i\,R\,dx = 0 \qquad \ldots (5)$$

where, w_i = weighting function

From equation (2), we know that,

$$w_1 = (x^2 - 2\,L\,x)$$

$$w_2 = (x^3 - 3\,L^2 x)$$

Substitute w_1, w_2 and R values in equation (5),

$$(5) \Rightarrow \quad \int_0^L (x^2 - 2\,L\,x)\,[A\,E\,(2\,a_2 + 6\,a_3 x) + a\,x]\,dx = 0 \qquad \ldots (6)$$

$$\int_0^L (x^3 - 3\,L^2 x)\,[A\,E\,(2\,a_2 + 6\,a_3 x) + a\,x]\,dx = 0 \qquad \ldots (7)$$

$$(6) \Rightarrow \quad \int_0^L (x^2 - 2\,L\,x)\,[A\,E\,(2\,a_2 + 6\,a_3 x) + a\,x]\,dx = 0$$

$$\int_0^L (x^2 - 2\,L\,x)\,[2\,a_2 A\,E + 6\,a_3 A\,E\,x + a\,x]\,dx = 0$$

$$\int_0^L [2\,a_2 A\,E\,x^2 + 6\,a_3 A\,E\,x^3 + a\,x^3 - 4\,a_2 A\,E\,L\,x - 12\,a_3 A\,E\,L\,x^2 - 2\,a\,L\,x^2]\,dx = 0$$

$$\Rightarrow \left[2\,a_2 AE\frac{x^3}{3} + 6\,a_3 AE\frac{x^4}{4} + a\frac{x^4}{4} - 4\,a_2 AE\,L\frac{x^2}{2} - 12\,a_3 AE\,L\frac{x^3}{3} - 2\,a\,L\frac{x^3}{3}\right]_0^L = 0$$

$$\Rightarrow \quad 2\,a_2 A\,E\frac{L^3}{3} + 6\,a_3 A\,E\frac{L^4}{4} + a\frac{L^4}{4} - 4\,a_2 A\,E\frac{L^3}{2} - 12\,a_3 A\,E\frac{L^4}{3} - 2\,a\frac{L^4}{3} = 0$$

$$\Rightarrow \quad \frac{2}{3} a_2 \text{AE} L^3 + \frac{3}{2} a_3 \text{A E} L^4 + \frac{a L^4}{4} - 2 a_2 \text{A E} L^3 - 4 a_3 \text{A E} L^4 - \frac{2}{3} a L^4 = 0$$

$$\Rightarrow \quad \text{AE} a_2 L^3 \left[\frac{2}{3} - 2\right] + a_3 \text{AE} L^4 \left[\frac{3}{2} - 4\right] + \frac{a L^4}{4} - \frac{2}{3} a L^4 = 0$$

$$\Rightarrow \quad \frac{-4}{3} \text{AE} L^3 a_2 - \frac{5}{2} \text{A E} L^4 a_3 = \left[\frac{2}{3} - \frac{1}{4}\right] a L^4$$

$$\Rightarrow \quad \frac{-4}{3} \text{A E} L^3 a_2 - \frac{5}{2} \text{A E} L^4 a_3 = \frac{5}{12} a L^4$$

$$\Rightarrow \quad \frac{4}{3} \text{A E} a_2 L^3 + \frac{5}{2} \text{A E} a_3 L^4 = \frac{-5}{12} a L^4 \qquad \ldots (8)$$

Equation (7),

$$\Rightarrow \quad \int_0^L (x^3 - 3 L^2 x) [\text{AE} (2 a_2 + 6 a_3 x) + a x] \, dx = 0$$

$$\Rightarrow \quad \int_0^L [x^3 - 3 L^2 x] [2 a_2 \text{A E} + 6 a_3 \text{A E} x + a x] \, dx = 0$$

$$\Rightarrow \quad \int_0^L [2 \text{A E} a_2 x^3 + 6 \text{A E} a_3 x^4 + a x^4 - 6 \text{A E} a_2 L^2 x - 18 \text{A E} a_3 L^2 x^2 - 3 a L^2 x^2] \, dx = 0$$

$$\Rightarrow \left[2\text{AE} a_2 \frac{x^4}{4} + 6\text{AE} a_3 \frac{x^5}{5} + \frac{a x^5}{5} - 6 \text{AE} a_2 L^2 \frac{x^2}{2} - 18\text{AE} a_3 L^2 \frac{x^3}{3} - 3 a L^2 \frac{x^3}{3} \right]_0^L = 0$$

$$\Rightarrow \left[\frac{1}{2} \text{AE} a_2 x^4 + \frac{6}{5} \text{AE} a_3 x^5 + \frac{1}{5} a x^5 - 3 \text{AE} a_2 L^2 x^2 - 6 \text{AE} a_3 L^2 x^3 - a L^2 x^3 \right]_0^L$$

$$\Rightarrow \frac{1}{2} \text{AE} a_2 L^4 + \frac{6}{5} \text{AE} a_3 L^5 + \frac{1}{5} a L^5 - 3 \text{AE} a_2 L^2 (L^2) - 6 \text{AE} a_3 L^2 (L^3) - a L^2 (L^3) = 0$$

$$\Rightarrow \frac{1}{2} \text{A E} a_2 L^4 + \frac{6}{5} \text{A E} a_3 L^5 + \frac{1}{5} a L^5 - 3 \text{A E} a_2 L^4 - 6 \text{A E} a_3 L^5 - a L^5 = 0$$

$$\Rightarrow \quad \text{AE} a_2 L^4 \left[\frac{1}{2} - 3\right] + \text{AE} a_3 L^5 \left[\frac{6}{5} - 6\right] + a L^5 \left[\frac{1}{5} - 1\right] = 0$$

$$\Rightarrow \quad \text{AE} a_2 L^4 \left[\frac{-5}{2}\right] - \frac{24}{5} \text{AE} a_3 L^5 = \frac{4}{5} a L^5$$

$$\Rightarrow \quad \frac{5}{2} \text{AE} a_2 L^4 + \frac{24}{5} \text{AE} a_3 L^5 = \frac{-4}{5} a L^5 \qquad \ldots (9)$$

Solving equations (8) and (9),

Equation (8) $\Rightarrow$ $$\frac{4}{3} \text{AE} a_2 L^3 + \frac{5}{2} \text{AE} a_3 L^4 = \frac{-5}{12} a L^4$$

Equation (9) $\Rightarrow$ $$\frac{5}{2} \text{AE} a_2 L^4 + \frac{24}{5} \text{AE} a_3 L^5 = \frac{-4}{5} a L^5$$

Multiplying equation (8) by $\frac{5}{2}$ L and equation (9) by $\frac{4}{3}$,

$$\frac{20}{6} \text{AE} a_2 L^4 + \frac{25}{4} \text{AE} a_3 L^5 = \frac{-25}{24} a L^5$$

$$\frac{20}{6} \text{AE} a_2 L^4 + \frac{96}{15} \text{AE} a_3 L^5 = \frac{-16}{15} a L^5$$

Subtracting,

$$\left(\frac{25}{4} - \frac{96}{15}\right) \text{AE} a_3 L^5 = \left(\frac{16}{15} - \frac{25}{24}\right) a L^5$$

$$\left(\frac{375 - 384}{60}\right) \text{AE} a_3 L^5 = \left(\frac{384 - 375}{360}\right) a L^5$$

$$\Rightarrow \frac{-9}{60} \text{AE} a_3 L^5 = \frac{9}{360} a L^5$$

$$\Rightarrow -0.15 \text{AE} a_3 = 0.025 a$$

$$\Rightarrow a_3 = -0.1666 \frac{a}{\text{A E}}$$

$$\Rightarrow \boxed{a_3 = \frac{-a}{6 \text{A E}}} \qquad \ldots (10)$$

Substituting a_3 value in equation (8),

$$\frac{4}{3} \text{AE} a_2 L^3 + \frac{5}{2} \text{AE} \left(\frac{-a}{6 \text{AE}}\right) L^4 = \frac{-5}{12} a L^4$$

$$\frac{4}{3} \text{AE} a_2 L^3 = \frac{-5}{12} a L^4 - \frac{5}{2} \text{AE} L^4 \left(\frac{-a}{6 \text{AE}}\right)$$

$$\frac{4}{3} \text{AE} a_2 L^3 = \frac{-5}{12} a L^4 + \frac{5}{12} a L^4$$

$$\frac{4}{3} \text{AE} a_2 L^3 = 0$$

$$\Rightarrow \boxed{a_2 = 0}$$

We know that, $R = -2a_1 + 50$

$$\frac{\partial R}{\partial a_1} = -2$$

Substitute R and $\frac{\partial R}{\partial a_1}$ values in equation (5),

$$\Rightarrow \quad \frac{\partial I}{\partial a_1} = \int_0^{10} (-2a_1 + 50)(-2)\,dx$$

The requirement is, $\frac{\partial I}{\partial a_1} = 0$

$$\Rightarrow \quad \int_0^{10} (-2a_1 + 50)(-2)\,dx = 0$$

$$\Rightarrow \quad \int_0^{10} (-2a_1 + 50)\,dx = 0$$

$$\Rightarrow \quad \int_0^{10} [-2a_1\,dx + 50\,dx] = 0$$

$$\Rightarrow \quad \Big[-2a_1 x + 50x\Big]_0^{10} = 0$$

$$\Rightarrow \quad -2a_1(10) + 50(10) - [0] = 0$$

$$\Rightarrow \quad -20a_1 + 500 = 0$$

$$\Rightarrow \quad -20a_1 = -500$$

$$\Rightarrow \quad \boxed{a_1 = 25} \qquad \ldots (6)$$

Therefore, the trial function becomes, $y = 25x(10 - x)$.

(iv) Galerkin's method: In this method, the trial function itself is considered as the weighting function, w_i.

$$\Rightarrow \quad \int_0^{10} w_i R\,dx = 0 \qquad \ldots (7)$$

Here, the trial function is, $y = w_i = a_1 x(10 - x)$.

Substitute w_i and R values in equation (7),

$$\Rightarrow \quad \int_0^{10} a_1 x(10 - x) \times (-2a_1 + 50)\,dx = 0$$

$$\Rightarrow \quad a_1 \int_0^{10} x(10 - x) \times (-2a_1 + 50)\,dx = 0$$

$$\Rightarrow \quad a_1 \int_0^{10} (10x - x^2)(-2a_1 + 50)\,dx = 0$$

$$\Rightarrow \quad a_1 \int_0^{10} [-20a_1 x + 500x + 2a_1 x^2 - 50x^2]\,dx = 0$$

$$\Rightarrow \quad a_1 \left[-20a_1 \frac{x^2}{2} + 500\frac{x^2}{2} + 2a_1 \frac{x^3}{3} - 50\frac{x^3}{3}\right]_0^{10} = 0$$

$$\Rightarrow \quad \frac{-20a_1}{2}[10^2 - 0] + \frac{500}{2}[10^2 - 0] + \frac{2a_1}{3}[10^3 - 0] - \frac{50}{3}[10^3 - 0] = 0$$

$$\Rightarrow \quad -10a_1[100] + 250[100] + \frac{2a_1}{3}[1000] - \frac{50}{3}[1000] = 0$$

$$\Rightarrow \quad -1000a_1 + 25{,}000 + 666.66a_1 - 16{,}666.66 = 0$$

$$\Rightarrow \quad -333.33a_1 = -8333.33$$

$$\Rightarrow \quad \boxed{a_1 = 25} \qquad \ldots (8)$$

The trial function is, $y = 25x(10 - x)$

From equations (3), (4), (6) and (8), we know that the value of parameter a_1 is same for all the four methods.

Result: Parameter, a_1 (for all the four methods) = 25

Example 1.14 ***The differential equation of a physical phenomenon is given by,***

$$\frac{d^2y}{dx^2} + 500x^2 = 0, \quad 0 \le x \le 1$$

Trial function, $y = a_1(x - x^4)$

Boundary conditions are, $y(0) = 0$

$y(1) = 0$

Calculate the value of the parameter a_1 by the following methods:

(i) Point collocation; (ii) Subdomain collocation; (iii) Least squares; (iv) Galerkin.

Given: Differential equation, $\frac{d^2y}{dx^2} + 500\,x^2 = 0,\ 0 \le x \le 1$ ... (1)

Trial function, $y = a_1\,(x - x^4)$

Boundary conditions are, $y(0) = 0$

$y(1) = 0$

To find: The value of parameter a_1 by,

(*i*) Point collocation method.

(*ii*) Subdomain collocation method.

(*iii*) Least squares method.

(*iv*) Galerkin's method.

☺ ***Solution:*** First we have to verify, whether the trial function satisfies the boundary conditions or not.

Trial function is, $y = a_1\,(x - x^4)$

When $x = 0,\ y = 0$

$x = 1,\ y = 0$

Hence it satisfies the boundary conditions.

(i) Point collocation method:

$$y = a_1\,(x - x^4)$$

$$\frac{dy}{dx} = a_1\,(1 - 4\,x^3)$$

$$\frac{d^2y}{dx^2} = a_1\,(0 - 12\,x^2)$$

$$\boxed{\frac{d^2y}{dx^2} = -12\,a_1\,x^2}$$

Substituting $\frac{d^2y}{dx^2}$ value in given differential equation (1),

$$\Rightarrow \boxed{\text{Residual, } R = -12\,a_1\,x^2 + 500\,x^2} \quad \text{... (2)}$$

In point collocation method, residuals are set to zero.

$$\Rightarrow R = -12\,a_1\,x^2 + 500\,x^2 = 0 \quad \text{... (3)}$$

In this problem, we have to find only one parameter, a_1. So, only one collocation point is needed. The point may be chosen between 0 and 1. Let us take $\frac{1}{2}$,

Substituting $x = \frac{1}{2}$ in equation (3),

$$\Rightarrow R = -12\,a_1\left(\frac{1}{2}\right)^2 + 500\left(\frac{1}{2}\right)^2 = 0$$

$$\Rightarrow -12\,a_1\left(\frac{1}{4}\right) + 500\left(\frac{1}{4}\right) = 0$$

$$\Rightarrow -3\,a_1 + 125 = 0$$

$$\Rightarrow \boxed{a_1 = 41.66} \quad \text{... (4)}$$

Hence the trial function is, $y = 41.66\,(x - x^4)$

(ii) Subdomain collocation method:

This method requires, $\int_0^1 R\,dx = 0$

Substitute R value,

$$\Rightarrow \int_0^1 (-12\,a_1\,x^2 + 500\,x^2)\,dx = 0$$

$$\Rightarrow -12\,a_1\left[\frac{x^3}{3}\right]_0^1 + 500\left[\frac{x^3}{3}\right]_0^1 = 0$$

$$\Rightarrow \frac{-12\,a_1}{3}[1 - 0] + \frac{500}{3}[1 - 0] = 0$$

$$\Rightarrow \frac{-12\,a_1}{3} + \frac{500}{3} = 0$$

$$\Rightarrow -12\,a_1 + 500 = 0$$

$$\Rightarrow -12\,a_1 = -500$$

$$\Rightarrow a_1 = \frac{500}{12}$$

$$\Rightarrow \boxed{a_1 = 41.66} \quad \text{... (5)}$$

Trial function is, $y = 41.66\,(x - x^4)$

(iii) Least squares method:

This method requires, $I = \int_0^1 R^2\,dx$

It can also be written as, $\frac{\partial I}{\partial a_1} = \int_0^1 R\,\frac{\partial R}{\partial a_1}\,dx$ ... (6)

We know that, $R = -12\,a_1 x^2 + 500\,x^2$

$$\frac{\partial R}{\partial a_1} = -12\,x^2$$

Substitute R and $\frac{\partial R}{\partial a_1}$ values in equation (6),

$$\Rightarrow \frac{\partial I}{\partial a_1} = \int_0^1 (-12\,a_1 x^2 + 500\,x^2)(-12\,x^2)\,dx$$

The requirement is, $\frac{\partial I}{\partial a_1} = 0$

$$\Rightarrow \int_0^1 (-12\,a_1 x^2 + 500\,x^2)(-12\,x^2)\,dx = 0$$

$$\Rightarrow \int_0^1 (144\,a_1 x^4 - 6000\,x^4)\,dx = 0$$

$$\Rightarrow 144\,a_1\left[\frac{x^5}{5}\right]_0^1 - 6000\left[\frac{x^5}{5}\right]_0^1 = 0$$

$$\Rightarrow \frac{144\,a_1}{5}[1-0] - \frac{6000}{5}[1-0] = 0$$

$$\Rightarrow 28.8\,a_1 = 1200$$

$$\Rightarrow \boxed{a_1 = 41.66} \quad \ldots (7)$$

The trial function is, $y = 41.66\,(x - x^4)$

(iv) Galerkin's method: In this method, the trial function itself is considered as the weighting function, w_i.

$$\Rightarrow \int_0^1 w_i\,R\,dx = 0 \quad \ldots (8)$$

Here, the trial function is $y = w_i = a_1\,(x - x^4)$

Substitute w_i and R values in equation (8).

$$\Rightarrow \int_0^1 a_1\,(x - x^4)(-12\,a_1 x^2 + 500\,x^2)\,dx = 0$$

$$\Rightarrow a_1 \int_0^1 (x - x^4)(-12\,a_1 x^2 + 500\,x^2)\,dx = 0$$

$$\Rightarrow a_1 \int_0^1 (-12\,a_1 x^3 + 500\,x^3 + 12\,a_1 x^6 - 500\,x^6)\,dx = 0$$

$$\Rightarrow a_1\left[-12\,a_1\left(\frac{x^4}{4}\right)_0^1 + 500\left(\frac{x^4}{4}\right)_0^1 + 12\,a_1\left(\frac{x^7}{7}\right)_0^1 - 500\left(\frac{x^7}{7}\right)_0^1\right] = 0$$

$$\Rightarrow \frac{-12\,a_1}{4}[1-0] + \frac{500}{4}(1-0) + \frac{12\,a_1}{7}(1-0) - \frac{500}{7}(1-0) = 0$$

$$\Rightarrow -3\,a_1 + 125 + 1.714\,a_1 - 71.428 = 0$$

$$\Rightarrow -1.286\,a_1 = -53.572$$

$$\Rightarrow \boxed{a_1 = 41.66} \quad \ldots (9)$$

Trial function is $y = 41.66\,(x - x^4)$

From equations (4), (5), (7) and (9), we know that the value of parameter a_1 is same for all the four methods.

Result: Parameter, a_1 (For all the four methods) = 41.66

Example 1.15 ***The differential equation of a physical phenomenon is given by***

$$\frac{d^2y}{dx^2} + 500\,x^2 = 0; \quad 0 \le x \le 1$$

By using the trial function, $y = a_1\,(x - x^3) + a_2\,(x - x^5)$, calculate the value of the parameters a_1 and a_2 by the following methods:

(i) Point collocation; (ii) Subdomain collocation; (iii) Least squares; (iv) Galerkin.

The boundary conditions are: $y(0) = 0$

$y(1) = 0$

Given: Differential equation, $\frac{d^2y}{dx^2} + 500\,x^2 = 0, \; 0 \le x \le 1 \quad \ldots (1)$

Trial function, $y = a_1\,(x - x^3) + a_2\,(x - x^5)$

Boundary conditions are: $y(0) = 0$

$y(1) = 0$

To find: The value of the parameters a_1 and a_2 by,

(*i*) Point collocation method.
(*ii*) Subdomain collocation method.
(*iii*) Least squares method.
(*iv*) Galerkin's method.

☺ ***Solution:*** First we have to verify, whether the trial function satisfies the boundary conditions or not.

The trial function is $y = a_1\,(x - x^3) + a_2\,(x - x^5)$

When $x = 0, \; y = 0$

$x = 1, \; y = 0$

Hence it satisfies the boundary conditions.

$$\Rightarrow \qquad a_2\left[0.6667 - 80\,l^2 + 200\,l^2 - 2 + 200\,l^2 - \frac{1600\,l^2}{3}\right] = -72000$$

Substitute $L = 50 \times 10^{-3}$ m (given)

$$\Rightarrow \; a_2\,[0.6667 - 0.2 + 0.5 - 2 + 0.5 - 1.3333] = -72000$$

$$\Rightarrow \; a_2\,[-1.8666] = -72000$$

$$\boxed{a_2 = 38572.80}$$

Galerkin solution, $T(x) = 300 + 38572.80\,(x^2 - 2\,L\,x)$

Result: Galerkin solution, $T(x) = 300 + 38572.80\,(x^2 - 2\,L\,x)$

Substitute, $\frac{d^4W}{dx^4}$, $\frac{d^4W}{dy^4}$ and $\frac{d^4W}{dx^2\,dy^2}$ in governing equation,

$$\Rightarrow \frac{E\,h^3}{12\,(1-\nu^2)}\left[\, a_1 \sin\frac{\pi x}{a}\left(\frac{\pi^4}{a^4}\right)\cdot \sin\left(\frac{\pi y}{b}\right) + 2\,a_1 \sin\frac{\pi x}{a}\,\sin\frac{\pi y}{b}\cdot\frac{\pi^4}{a^2\,b^2} + a_1 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\cdot\frac{\pi^4}{b^4}\,\right] - q_0 = 0 \qquad \ldots (4)$$

Take residual, $$R = \frac{E\,h^3}{12\,(1-\nu^2)}\left[\, a_1 \sin\frac{\pi x}{a}\,\sin\frac{\pi y}{b}\left(\frac{\pi^4}{a^4}+\frac{2\,\pi^4}{a^2\,b^2}+\frac{\pi^4}{b^4}\right)\right] - q_0$$

Using Galerkin's method, $$\int_0^b\int_0^a W(x,y)\,R\,dx\,dy = 0$$

$$\int_0^b\int_0^a \sin\frac{\pi x}{a}\,\sin\frac{\pi y}{b}\left[\,\frac{E\,h^3}{12\,(1-\nu^2)}\,a_1 \sin\frac{\pi x}{a}\,\sin\frac{\pi y}{b}\left(\frac{\pi^4}{a^4}+\frac{\pi^4}{b^4}+\frac{2\,\pi^4}{a^2\,b^2}\right) - q_0\,\right] dx\,dy = 0$$

$$\Rightarrow \frac{E\,h^3}{12\,(1-\nu^2)}\,a_1\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2 \int_0^b\int_0^a \sin^2\frac{\pi x}{a}\,\sin^2\frac{\pi y}{b}\,dx\,dy - \int_0^b\int_0^a q_0 \sin\frac{\pi x}{a}\,\sin\frac{\pi y}{b}\,dx\,dy = 0$$

$$\Rightarrow \frac{E\,h^3}{12\,(1-\nu^2)}\,a_1\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2 \int_0^b\int_0^a \left(\frac{1+\cos\frac{2\,\pi x}{a}}{2}\right)\sin^2\frac{\pi y}{b}\,dx\,dy = q_0\int_0^b\int_0^a \sin\frac{\pi x}{a}\,\sin\frac{\pi y}{b}\,dx\,dy$$

$$\Rightarrow \frac{E\,h^3}{12\times 2\,(1-\nu^2)}\,a_1\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2 \int_0^b \left(x+\sin\frac{2\,\pi x}{a}\cdot\frac{a}{2\,\pi}\right)_0^a \sin^2\frac{\pi y}{b}\,dy = q_0\int_0^b\left(-\cos\frac{\pi x}{a}\cdot\frac{a}{\pi}\right)_0^a \sin\frac{\pi y}{b}\,dy$$

$$\Rightarrow \frac{E\,h^3}{12\times 2\,(1-\nu^2)}\,a_1\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2 \int_0^b a\,\sin^2\frac{\pi y}{b}\,dy = q_0\int_0^b\left[-(-1)\cdot\frac{a}{\pi}-(-1)\frac{a}{\pi}\right]\cdot\sin\frac{\pi y}{b}\,dy$$

$[\because \sin 2\pi = 0;\ \cos\pi = -1]$

$$\Rightarrow \frac{E\,h^3}{12\times 2\,(1-\nu^2)}\,a_1\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2 \int_0^b a\,\sin^2\frac{\pi y}{b}\,dy = q_0\left(\frac{2\,a}{\pi}\right)\int_0^b \sin\frac{\pi y}{b}\,dy$$

$$\Rightarrow \frac{E\,h^3}{12\times 2\,(1-\nu^2)}\,a_1\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2 a\int_0^b \sin^2\frac{\pi y}{b}\,dy = 2\,q_0\,\frac{a}{\pi}\int_0^b \sin\frac{\pi y}{b}\,dy$$

$$\Rightarrow \frac{E\,h^3}{12\times 2\,(1-\nu^2)}\,a_1\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2 \int_0^b\left(\frac{1+\cos\frac{2\,\pi y}{b}}{2}\right)dy = \frac{2\,q_0}{\pi}\left(-\cos\frac{\pi y}{b}\right)_0^b\cdot\frac{b}{\pi}$$

$$\Rightarrow \frac{E\,h^3}{12\times 4\,(1-\nu^2)}\,a_1\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]\left[y+\sin\frac{2\,\pi y}{b}\cdot\frac{b}{2\,\pi}\right]_0^b = \frac{2\,q_0}{\pi}\left[\frac{b}{\pi}+\frac{b}{\pi}\right]$$

$$\Rightarrow \frac{E\,h^3}{12\times 4\,(1-\nu^2)}\,a_1\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2 b = \frac{2\,q_0}{\pi}\cdot\frac{2\,b}{\pi}$$

$$\Rightarrow \boxed{a_1 = \frac{16\,q_0}{\pi^2}\,\frac{12\,(1-\nu^2)}{E\,h^3}\left[\frac{1}{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\right]^2} \qquad \ldots (5)$$

If square plate, $a = b$ $\qquad \ldots (6)$

Substitute the equation (6) in equation (5),

$$a_1 = \frac{16\,q_0}{\pi^2}\,\frac{12\,(1-\nu^2)}{E\,h^3}\left[\frac{1}{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{a}\right)^2}\right]^2$$

$$a_1 = \frac{16\,q_0}{\pi^2}\,\frac{12\,(1-\nu^2)}{E\,h^3}\left[\frac{1}{4\left(\frac{\pi^4}{a^4}\right)}\right]$$

$$a_1 = \frac{16\,q_0}{\pi^2}\,\frac{12\,(1-\nu^2)}{E\,h^3}\left[\frac{a^4}{4\,\pi^4}\right]$$

$$\boxed{a_1 = \frac{4\,q_0\,a^4}{\pi^6}\left[\frac{12\,(1-\nu^2)}{E\,h^3}\right]} \qquad \ldots (7)$$

Result: Parameter, by using Galerkin technique for rectangular plate,

$$\text{Parameter, } a_1 \text{ (for rectangular)} = \left(\frac{16\,q_0}{\pi^2}\right)\left(\frac{12\,(1-\nu^2)}{E\,h^3}\right)\left[\frac{1}{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\right]^2$$

$$\text{Parameter, } a_1 \text{ (for square)} = \frac{4\,q_0\,a^4}{\pi^6}\left[\frac{12\,(1-\nu^2)}{E\,h^3}\right]$$

1.15. VARIATIONAL (WEAK) FORM OF THE WEIGHTED RESIDUAL STATEMENT

We know that the general weighted residual statement is,

$$\int w\,R\,dx = 0 \qquad \ldots (1.15)$$

In this variational method, integration is carried out by parts. It reduces the continuity requirement on the trial function assumed in the solution. So, it is referred to as the weak form. In this method, it is possible to have a wider choice of trial functions.

1.16. COMPARISON OF DIFFERENTIAL EQUATION, WEIGHTED RESIDUAL STATEMENT AND WEAK FORMULATION OF WEIGHTED RESIDUAL STATEMENT

1.16.1. Differential Equation

Consider a uniform rod subjected to uniform axial load q_0 as shown in Fig.1.26.

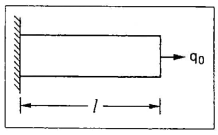


Fig. 1.26. Uniform rod

The deformation of the bar is governed by the differential equation,

$$A\,E\,\frac{d^2u}{dx^2} + q_0 = 0 \qquad \ldots (1.16)$$

With the boundary conditions,

$$u(0) = 0$$

$$A\,E\,\frac{du}{dx}\bigg|_{x=l} = P_l \qquad \ldots (1.17)$$

1.16.2. Weighted Residual Statement

In order to find the solution for the above mentioned problem, the weighted residual statement can be developed as follows:

$$\int_0^l w(x)\left[A\,E\,\frac{d^2u}{dx^2} + q_0\right]dx = 0 \qquad \ldots (1.18)$$

With the boundary conditions,

$$u(0) = 0$$

$$A\,E\,\frac{du}{dx}\bigg|_{x=l} = P_l \qquad \ldots (1.19)$$

1.16.3. Observations on the Weighted Residual Statement

- ✓ Weighted residual statement can be developed for any form of differential equations like linear, non-linear, ordinary, partial, *etc.*
- ✓ The weighted residual statement is developed only for differential equation and it is not suitable for boundary conditions.
- ✓ The trial solution should satisfy all the boundary conditions and it should be differentiable as many times as needed in the original differential equation.

1.16.4. Weak Form of Weighted Residual Statement

By performing integration by parts, the weak form of weighted residual statement of the above mentioned problem is obtained as follows:

$$\left[w(x)\,A\,E\,\frac{du}{dx}\right]_0^l - \int_0^l A\,E\,\frac{du}{dx}\cdot\frac{dw}{dx}\cdot dx + \int_0^l w(x)\,q\,dx = 0 \qquad \ldots (1.20)$$

With the boundary conditions,

$$u(0) = 0$$

$$A\,E\,\frac{du}{dx}\bigg|_{x=l} = P_l$$

In, principle of virtual work, $\delta U = \delta H$

$$\therefore \delta\pi = 0$$

Hence, we can conclude that a deformable body is in equilibrium when the potential energy is having stationary value.

Hence, the principal of minimum potential energy states *"Among all the displacement equations that internal compatibility and the boundary condition those that also satisfy the equations of equilibrium make the potential energy a minimum is a stable system."*

1.19. SOLVED PROBLEMS – POTENTIAL ENERGY APPROACH

Example 1.23 ***The spring assembly is shown in Fig.(i). Assemble the finite element equation by using direct approach and potential energy approach.***

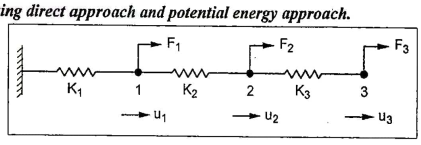


Fig. (i).

Given:

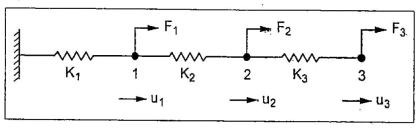


Fig. (ii).

To find: Global stiffness matrix for the spring system.

☺ ***Solution:*** Consider the free body diagram of nodes 1, 2 and 3 as shown in Fig.(iii). Let the displacement of nodes be u_1, u_2 and u_3. The extension of spring 1, 2 and 3 are,

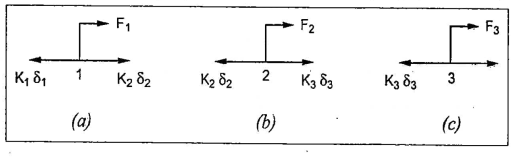


Fig. (iii).

We know that, Displacement,

$$\delta_1 = u_1, \quad \delta_2 = u_2 - u_1, \quad \text{and } \delta_3 = u_3 - u_2 \qquad \ldots(1)$$

The equilibrium equations are (from Fig.(iii))

$$-k_1\,\delta_1 + k_2\,\delta_2 + F_1 = 0 \qquad \ldots(2)$$

$$-k_2\,\delta_2 + k_3\,\delta_3 + F_2 = 0 \qquad \ldots(3)$$

$$-k_3\,\delta_3 + F_3 = 0 \qquad \ldots(4)$$

Substitute δ_1, δ_2 and δ_3 values in equations (2), (3) and (4).

Equation (2) ⇒

$$-k_1\,u_1 + k_2\,(u_2 - u_1) = -F_1$$

$$k_1\,u_1 - k_2\,(u_2 - u_1) = F_1$$

$$(k_1 + k_2)\,u_1 - k_2\,u_2 = F_1 \qquad \ldots(5)$$

Equation (3) ⇒

$$-k_2\,(u_2 - u_1) + k_3\,(u_3 - u_2) = -F_2$$

$$k_2\,(u_2 - u_1) - k_3\,(u_3 - u_2) = F_2$$

$$-k_2\,u_1 + (k_2 + k_3)\,u_2 - k_3\,u_3 = F_2 \qquad \ldots(6)$$

Equation (4) ⇒

$$-k_3\,(u_3 - u_2) = -F_3$$

$$k_3\,(u_3 - u_2) = F_3$$

$$-k_3\,u_2 + k_3\,u_3 = F_3 \qquad \ldots(7)$$

Arranging equations (5), (6) and (7) in matrix form,

$$\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad \ldots(8)$$

Now, let us see the potential energy approach. Total potential energy in the system is,

$$\pi = \frac{1}{2}\,k_1\,\delta_1^2 + \frac{1}{2}\,k_2\,\delta_2^2 + \frac{1}{2}\,k_3\,\delta_3^2 - F_1\,u_1 - F_2\,u_2 - F_3\,u_3$$

$$= \frac{1}{2}\,k_1\,u_1^2 + \frac{1}{2}\,k_2\,(u_2 - u_1)^2 + \frac{1}{2}\,k_3\,(u_3 - u_2)^2 - F_1\,u_1 - F_2\,u_2 - F_3\,u_3$$

Apply, $\dfrac{\partial\pi}{\partial u_1} = 0$

$$\Rightarrow \quad k_1\,u_1 + k_2\,(u_2 - u_1)(-1) - F_1 = 0$$

$$k_1\,u_1 - k_2\,(u_2 - u_1) - F_1 = 0$$

$$(k_1 + k_2)\,u_1 - k_2\,u_2 = F_1 \qquad \ldots(9)$$

Apply, $\frac{\partial \pi}{\partial u_2} = 0$

$$\Rightarrow \quad k_2 (u_2 - u_1) + k_3 (u_3 - u_2)(-1) = F_2$$

$$-k_2 u_1 + (k_2 + k_3) u_2 - k_3 u_3 = F_2 \quad \ldots(10)$$

Apply, $\frac{\partial \pi}{\partial u_3} = 0$

$$\Rightarrow \quad k_3 (u_3 - u_2) - F_3 = 0$$

$$-k_3 u_2 + k_3 u_3 = F_3 \quad \ldots(11)$$

Equation (9), (10) and (11) in matrix form,

$$\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \ldots(12)$$

Result: Finite element equation

$$\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Example 1.24 ***Determine the displacements of nodes 1 and 2 in the spring system shown in Fig.(i). Use minimum of potential energy principle to assemble equations of equilibrium.***

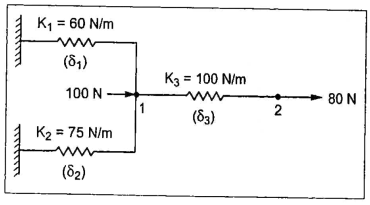


Fig. (i).

Given:

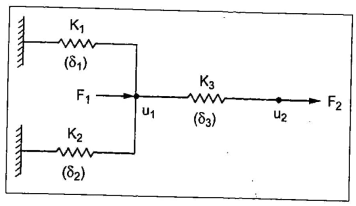


Fig. (ii).

$k_1 = 60$ N/m, $\quad F_1 = 100$ N

$k_2 = 75$ N/m, $\quad F_2 = 80$ N

$k_3 = 100$ N/m

To find: Displacements of nodes 1 and 2.

☺ ***Solution:*** Let u_1 and u_2 be the displacements of nodes 1 and 2. Then the extensions of springs are,

$$\delta_1 = u_1, \quad \delta_2 = u_1, \quad \delta_3 = u_2 - u_1$$

We know that,

Minimum of potential energy principle,

$$\pi = \text{Strain energy} - \text{Work done}$$

$$\pi = \frac{1}{2} k_1 \delta_1^2 + \frac{1}{2} k_2 \delta_2^2 + \frac{1}{2} k_3 \delta_3^2 - 100\, u_1 - 80\, u_2$$

$$\pi = \frac{1}{2} k_1 u_1^2 + \frac{1}{2} k_2 u_1^2 + \frac{1}{2} k_3 (u_2 - u_1)^2 - 100\, u_1 - 80\, u_2 \quad \ldots(1)$$

$\therefore$ Now $\frac{\partial \pi}{\partial u_1} = 0$

$$\Rightarrow \quad k_1 u_1 + k_2 u_1 + k_3 (u_2 - u_1)(-1) - 100 = 0$$

$$\Rightarrow \quad (k_1 + k_2 + k_3) u_1 - k_3 u_2 = 100 \quad \ldots(2)$$

Similarly, $\frac{\partial \pi}{\partial u_2} = 0 \Rightarrow \quad k_3 (u_2 - u_1) - 80 = 0$

$$-k_3 u_1 + k_3 u_2 = 80 \quad \ldots(3)$$

Arranging equation (2) and (3) in matrix form,

$$\begin{bmatrix} k_1+k_2+k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 80 \end{Bmatrix}$$

Substituting the values of k_1, k_2 and k_3, we get,

$$\begin{bmatrix} 60+75+100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 80 \end{Bmatrix}$$

$$\begin{bmatrix} 235 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 80 \end{Bmatrix}$$

$$235\,u_1 - 100\,u_2 = 100 \quad \ldots(4)$$

$$-100\,u_1 + 100\,u_2 = 80 \quad \ldots(5)$$

$$135\,u_1 = 180$$

$$u_1 = \frac{180}{135}$$

$$\boxed{u_1 = 1.333}$$

Substitute the u_1 – value in equation (5)

$$-100\,(1.333) + 100\,u_2 = 80$$

$$\boxed{u_2 = 2.133}$$

Result: Displacements of nodes,

$$u_1 = 1.333 \text{ m}$$

$$u_2 = 2.133 \text{ m}$$

1.20. RAYLEIGH-RITZ METHOD (VARIATIONAL APPROACH)

1.20.1. Introduction

- ✓ Rayleigh-Ritz method is a integral approach method which is useful for solving complex structural problems, encountered in finite element analysis. This method is possible only if a suitable functional is available, otherwise Galerkin's method of weighted residual is used. By using this method stiffness matrices and consistent load vector can be assembled easily. This method is mostly used for solving solid mechanics problems.
- ✓ The phrase "Variational methods" refers to methods that make use of variational principles, such as the principles of virtual work and the principle of minimum potential energy in solid and structural mechanics, to determine the approximate solutions of the problems.

- ✓ In Rayleigh-Ritz method for continuous system we deal with the following functional.

$$\text{Potential energy, } \pi = \int_{x_1}^{x_2} f(y, y', y'')\, dx \quad \ldots(1.26)$$

- ✓ In our terminology, a functional is an integral expression that implicitly contains the governing differential equations for a particular problem.
- ✓ Total potential energy of the structure is given by,

$$\pi = \begin{Bmatrix} \text{Internal} \\ \text{potential} \\ \text{energy} \end{Bmatrix} - \begin{Bmatrix} \text{External} \\ \text{potential} \\ \text{energy} \end{Bmatrix}$$

$$= \text{Strain energy} - \text{Work done by external forces}$$

$$\pi = U - H$$

- ✓ In this method, the approximating functions must satisfy the boundary conditions and should be easy to use. Polynomials are generally used and sometimes sine and cosine terms are also used as approximating function.
- ✓ In general any exact function can be represented as a polynomial or trigonometric series with undetermined constants as shown below.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots$$

or

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} + \ldots\ldots$$

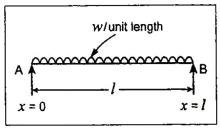
The constants a_0, a_1, a_2 are unknowns known as Ritz parameters of the curve. When the parameters are infinite, the particular polynomial tends to match the exact value. So, the accuracy depends upon the number of parameters chosen.

- ✓ The following two conditions must be fulfilled by the approximating function.
  1. It should satisfy the geometric boundary conditions.
  2. The function must have atleast one Ritz parameter.
- ✓ In general, a Rayleigh-Ritz solution is rarely exact except in some special simple cases, but it becomes more accurate with the use of more parameters.
- ✓ This method can be understood clearly by solving the following examples.

1.20.2. Solved Problems (on Rayleigh-Ritz Method)

Example 1.25 ***A simply supported beam subjected to uniformly distributed load over entire span. Determine the bending moment and deflection at midspan by using Rayleigh-Ritz method and compare with exact solutions.***

Given:



To find: 1. Deflection and Bending moment at midspan.

2. Compare with exact solutions.

☺ ***Solution:*** We know that, for simply supported beam, the Fourier series,

$$y = \sum_{n=1,3}^{\infty} a \sin\frac{n\pi x}{l} \text{ is the approximating function.}$$

To make this series more simple let us consider only two terms.

$$\text{Deflection, } y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} \qquad \ldots (1)$$

where, a_1, a_2 are Ritz parameters.

We know that,

$$\text{Total potential energy of the beam, } \pi = U - H \qquad \ldots (2)$$

where, U → Strain energy.

H → Work done by external force.

The strain energy, U, of the beam due to bending is given by,

$$U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad \ldots (3)$$

$$\frac{dy}{dx} = a_1 \cos\frac{\pi x}{l} \times \left(\frac{\pi}{l}\right) + a_2 \cos\frac{3\pi x}{l}\left(\frac{3\pi}{l}\right)$$

$$\frac{dy}{dx} = \frac{a_1\pi}{l}\cos\frac{\pi x}{l} + \frac{a_2 3\pi}{l}\cos\frac{3\pi x}{l}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a_1\pi}{l}\sin\frac{\pi x}{l} \times \frac{\pi}{l} - a_2\frac{3\pi}{l}\sin\frac{3\pi x}{l} \times \frac{3\pi}{l}$$

$$= \frac{-\pi^2 a_1}{l^2}\sin\frac{\pi x}{l} - a_2\frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}$$

$$\frac{d^2y}{dx^2} = \left[-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{a_2 9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right] \qquad \ldots (4)$$

Substituting $\frac{d^2y}{dx^2}$ value in equation (3),

$$\Rightarrow U = \frac{EI}{2}\int_0^l \left[-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{a_2 9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]^2 dx$$

$$= \frac{EI}{2}\int_0^l \left[\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{a_2 9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]^2 dx$$

$$= \frac{EI}{2}\times\frac{\pi^4}{l^4}\int_0^l \left[a_1\sin\frac{\pi x}{l} + 9a_2\sin\frac{3\pi x}{l}\right]^2 dx$$

$$U = \frac{EI}{2}\times\frac{\pi^4}{l^4}\int_0^l \left[a_1^2\sin^2\frac{\pi x}{l} + 81a_2^2\sin^2\frac{3\pi x}{l} + 2a_1\sin\frac{\pi x}{l}\, 9a_2\sin\frac{3\pi x}{l}\right]dx$$

$$[\because (a+b)^2 = a^2 + b^2 + 2ab]$$

$$U = \frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l \left[a_1^2\sin^2\frac{\pi x}{l} + 81a_2^2\sin^2\frac{3\pi x}{l} + 18a_1a_2\sin\frac{\pi x}{l}\sin\frac{3\pi x}{l}\right]dx \qquad \ldots (5)$$

$$\int_0^l a_1^2\sin^2\frac{\pi x}{l}dx = a_1^2\int_0^l \frac{1}{2}\left(1 - \cos\frac{2\pi x}{l}\right)dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= \frac{a_1^2}{2}\int_0^l \left(1 - \cos\frac{2\pi x}{l}\right)dx$$

$$= \frac{a_1^2}{2}\left[\int_0^l dx - \int_0^l \cos\frac{2\pi x}{l}dx\right]$$

$$= \frac{a_1^2}{2}\left[(x)_0^l - \left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}} \right)_0^l \right]$$

$$= \frac{a_1^2}{2}\left[l - 0 - \frac{l}{2\pi}\left(\sin\frac{2\pi l}{l} - \sin 0 \right) \right]$$

$$= \frac{a_1^2}{2}\left[l - \frac{l}{2\pi}(0-0) \right] = \frac{a_1^2 l}{2} \qquad [\because \sin 2\pi = 0;\ \sin 0 = 0]$$

$$\boxed{\int_0^l a_1^2 \sin^2\frac{\pi x}{l}\,dx = \frac{a_1^2 l}{2}} \qquad \ldots (6)$$

Similarly,

$$\int_0^l 81\, a_2^2 \sin^2\frac{3\pi x}{l} = 81\, a_2^2 \int_0^l \frac{1}{2}\left(1 - \cos\frac{6\pi x}{l} \right) dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2} \right]$$

$$= \frac{81\, a_2^2}{2}\left[\int_0^l dx - \int_0^l \cos\frac{6\pi x}{l}\,dx \right]$$

$$= \frac{81\, a_2^2}{2}\left[(x)_0^l - \left(\frac{\sin\frac{6\pi x}{l}}{\frac{6\pi}{l}} \right)_0^l \right]$$

$$= \frac{81\, a_2^2}{2}\left[l - 0 - \frac{l}{6\pi}\left(\sin\frac{6\pi l}{l} - \sin 0 \right) \right]$$

$$= \frac{81\, a_2^2}{2}\left[l - \frac{l}{6\pi}(\sin 6\pi - \sin 0) \right]$$

$$= \frac{81\, a_2^2}{2}\left[l - 0 \right] \qquad [\because \sin 6\pi = 0;\ \sin 0 = 0]$$

$$\Rightarrow \boxed{\int_0^l 81\, a_2^2 \sin^2\frac{3\pi x}{l}\,dx = \frac{81\, a_2^2 l}{2}} \qquad \ldots (7)$$

$$\int_0^l 18\, a_1 a_2 \sin\frac{\pi x}{l}\sin\frac{3\pi x}{l} = 18\, a_1 a_2 \int_0^l \sin\frac{\pi x}{l}\sin\frac{3\pi x}{l}$$

$$= 18\, a_1 a_2 \int_0^l \sin\frac{3\pi x}{l}\sin\frac{\pi x}{l}$$

$$= 18\, a_1 a_2 \int_0^l \frac{1}{2}\left(\cos\frac{2\pi x}{l} - \cos\frac{4\pi x}{l} \right) dx$$

$$\left[\because \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2} \right]$$

$$= \frac{18\, a_1 a_2}{2}\left[\int_0^l \cos\frac{2\pi x}{l}\,dx - \int_0^l \cos\frac{4\pi x}{l}\,dx \right]$$

$$= \frac{18\, a_1 a_2}{2}\left[\left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}} \right)_0^l - \left(\frac{\sin\frac{4\pi x}{l}}{\frac{4\pi}{l}} \right)_0^l \right]$$

$$= 9\, a_1 a_2\, [0 - 0] = 0 \qquad [\because \sin 2\pi = 0;\ \sin 4\pi = 0;\ \sin 0 = 0]$$

$$\Rightarrow \boxed{\int_0^l 18\, a_1 a_2 \sin\frac{\pi x}{l}\sin\frac{3\pi x}{l} = 0} \qquad \ldots (8)$$

Substitute (6), (7) and (8) in equation (5),

$$(5) \Rightarrow \qquad U = \frac{EI}{2}\frac{\pi^4}{l^4}\left[\frac{a_1^2 l}{2} + \frac{81\, a_2^2 l}{2} + 0 \right]$$

$$U = \frac{EI\,\pi^4 l}{4\, l^4}\,[a_1^2 + 81\, a_2^2]$$

$$\boxed{\text{Strain energy, } U = \frac{EI\,\pi^4}{4\, l^3}\,[a_1^2 + 81\, a_2^2]} \qquad \ldots (9)$$

We know that,

$$\text{Work done by external force, } H = \int_0^l \omega\, y\, dx = \int_0^l \omega\left(a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} \right) dx$$

$$= \omega \int_0^l \left(a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} \right) dx$$

$$= \omega \left[a_1 \int_0^l \sin\frac{\pi x}{l} dx + a_2 \int_0^l \sin\frac{3\pi x}{l} dx \right]$$

$$= \omega \left[a_1 \left(\frac{-\cos\frac{\pi x}{l}}{\frac{\pi}{l}} \right)_0^l + a_2 \left(\frac{-\cos\frac{3\pi x}{l}}{\frac{3\pi}{l}} \right)_0^l \right]$$

$$= \omega \left[\frac{-a_1 l}{\pi} \left(\cos\frac{\pi x}{l} \right)_0^l - \frac{a_2 l}{3\pi} \left(\cos\frac{3\pi x}{l} \right)_0^l \right]$$

$$= \omega \left[\frac{-a_1 l}{\pi} [(-1) - 1] - \frac{a_2 l}{3\pi} (-1 - 1) \right]$$

$$= \omega \left[\frac{2 a_1 l}{\pi} + \frac{2 a_2 l}{3\pi} \right] \qquad [\because \cos 0 = 1;\ \cos\pi = -1;\ \cos 3\pi = -1]$$

$$= \frac{2\omega l}{\pi} \left[a_1 + \frac{a_2}{3} \right]$$

$$\boxed{H = \frac{2\omega l}{\pi} \left(a_1 + \frac{a_2}{3} \right)} \qquad \ldots (10)$$

Substitute (9) and (10) values in equation (2).

$$(2) \Rightarrow \pi = U - H$$

$$\Rightarrow \pi = \frac{E I \pi^4}{4 l^3} (a_1^2 + 81 a_2^2) - \frac{2 \omega l}{\pi} \left(a_1 + \frac{a_2}{3} \right) \qquad \ldots (11)$$

For stationary value of π, the following conditions must be satisfied.

$$\frac{\partial \pi}{\partial a_1} = 0 \text{ and } \frac{\partial \pi}{\partial a_2} = 0$$

$$\Rightarrow \frac{\partial \pi}{\partial a_1} = \frac{E I \pi^4}{4 l^3} (2 a_1) - \frac{2\omega l}{\pi} = 0$$

$$\Rightarrow \frac{EI \pi^4}{4 l^3} \times 2 a_1 = \frac{2\omega l}{\pi}$$

$$\Rightarrow \boxed{a_1 = \frac{4\omega l^4}{EI\pi^5}}$$

Similarly, $$\frac{\partial \pi}{\partial a_2} = \frac{EI \pi^4}{4 l^3} (162 a_2) - \frac{2\omega l}{\pi} \left(\frac{1}{3} \right) = 0$$

$$\Rightarrow \frac{EI \pi^4}{4 l^3} (162 a_2) = \frac{2\omega l}{\pi} \left(\frac{1}{3} \right)$$

$$\Rightarrow a_2 = \frac{2\omega l}{3\pi} \times \frac{4 l^3}{162 EI \pi^4} = \frac{4 \omega l^4}{243 EI \pi^5}$$

$$\boxed{a_2 = \frac{4\omega l^4}{243 EI \pi^5}}$$

We know that, $$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

Substituting a_1 and a_2 values,

$$\Rightarrow \boxed{y = \frac{4\omega l^4}{EI \pi^5} \sin\frac{\pi x}{l} + \frac{4\omega l^4}{243 EI \pi^5} \sin\frac{3\pi x}{l}} \qquad \ldots (12)$$

We know that, maximum deflection occurs at $x = \frac{l}{2}$.

Substitute $x = \frac{l}{2}$ in equation (12),

$$\Rightarrow y_{max} = \frac{4\omega l^4}{EI \pi^5} \sin\frac{\pi \times \frac{l}{2}}{l} + \frac{4\omega l^4}{243 EI \pi^5} \sin\frac{3\pi \frac{l}{2}}{l}$$

$$\Rightarrow y_{max} = \frac{4\omega l^4}{EI \pi^5} \sin\frac{\pi}{2} + \frac{4\omega l^4}{243 EI \pi^5} \sin\frac{3\pi}{2}$$

$$y_{max} = \frac{4\omega l^4}{EI \pi^5} - \frac{4\omega l^4}{243 EI \pi^5}$$

$$= \frac{4\omega l^4}{EI \pi^5} \left[1 - \frac{1}{243} \right] \qquad \left[\because \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1 \right]$$

$$= \frac{4\omega l^4}{EI \pi^5} (0.9958) = \frac{3.98 \omega l^4}{EI \pi^5}$$

$$\Rightarrow \boxed{y_{max} = 0.0130 \frac{\omega l^4}{EI}} \qquad \ldots (13)$$

We know that, simply supported beam subjected to uniformly distributed load, maximum deflection is,

$$y_{max} = \frac{5}{384}\frac{\omega\, l^4}{EI}$$

$$\boxed{y_{max} = 0.0130\,\frac{\omega\, l^4}{EI}} \qquad \ldots (14)$$

From equations (13) and (14), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are same.

Bending Moment at Mid span

We know that,

$$\text{Bending moment, } M = EI\,\frac{d^2y}{dx^2} \qquad \ldots (15)$$

From equation (4), we know

$$\frac{d^2y}{dx^2} = -\left[\frac{a_1\,\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{a_2\,9\,\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]$$

Substituting a_1 and a_2 values,

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{4\omega\, l^4}{EI\,\pi^5}\times\frac{\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{4\omega\, l^4}{243\,EI\,\pi^5}\times\frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]$$

Maximum bending occurs at $x = \frac{l}{2}$.

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{4\omega\, l^4}{EI\,\pi^5}\times\frac{\pi^2}{l^2}\sin\frac{\pi\frac{l}{2}}{l} + \frac{4\omega\, l^4}{243\,EI\,\pi^5}\times\frac{9\pi^2}{l^2}\sin\frac{3\pi\frac{l}{2}}{l}\right]$$

$$= -\left[\frac{4\omega\, l^4}{EI\,\pi^5}\times\frac{\pi^2}{l^2}\sin\frac{\pi}{2} + \frac{4\omega\, l^4}{243\,EI\,\pi^5}\times\frac{9\pi^2}{l^2}\sin\frac{3\pi}{2}\right]$$

$$= -\left[\frac{4\omega\, l^4}{EI\,\pi^5}\frac{\pi^2}{l^2}(1) + \frac{4\omega\, l^4}{243\,EI\,\pi^5}\times\frac{9\,\pi^2}{l^2}(-1)\right]$$

$$\left[\because \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1\right]$$

$$= -\left[\frac{4\omega\, l^2\,\pi^2}{EI\,\pi^5} - \frac{36\,\omega\, l^2\,\pi^2}{243\,EI\,\pi^5}\right]$$

$$= -\frac{4\omega\, l^2}{EI\,\pi^3} + \frac{36\,\omega\, l^2}{243\,EI\,\pi^3}$$

$$= -\frac{4\omega\, l^2}{EI\,\pi^3} + \frac{0.148\,\omega\, l^2}{EI\,\pi^3} = -3.852\,\frac{\omega\, l^2}{EI\,\pi^3}$$

$$\boxed{\frac{d^2y}{dx^2} = -0.124\,\frac{\omega\, l^2}{EI}}$$

Substituting $\frac{d^2y}{dx^2}$ value in bending moment equation,

$$(15) \Rightarrow \quad M_{centre} = EI \times -(0.124)\,\frac{\omega\, l^2}{EI}$$

$$\Rightarrow \quad \boxed{M_{centre} = -0.124\,\omega\, l^2} \qquad \ldots (16)$$

[Negative sign indicates downward load]

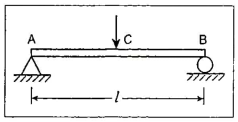
We know that, for simply supported beam subjected to uniformly distributed load, maximum bending moment is,

$$M_{centre} = \frac{\omega\, l^2}{8}$$

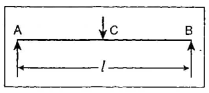
$$\boxed{M_{centre} = 0.125\,\omega\, l^2} \qquad \ldots (17)$$

From equation (16) and (17), we know that, exact solutoin and solution obtained by using Rayleigh-Ritz method are almost same. In order to get accurate result, more terms in Fourier series should be taken.

Example 1.26 ***A beam AB of span 'l' simply supported at ends and carrying a concentrated load W at the centre 'C' as shown in Fig. Determine the deflection at midspan by using Rayleigh-Ritz method and compare with exact solutions.***



Given:



To find: Deflection at midspan, y_{max}.

☺ ***Solution:*** From example (1.25), we know that,

$$\text{Deflection, } y = a_1\sin\frac{\pi x}{l} + a_2\sin\frac{3\pi x}{l} \qquad \ldots (1)$$

Total potential energy of the beam is given by,

$$\pi = U - H \qquad \dots (2)$$

where, $U \rightarrow$ Strain energy.

$H \rightarrow$ Work done by external force.

The strain energy U of the beam due to loading is given by,

$$U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad \dots (3)$$

From equation (9) in previous example problem (1.18), we know that,

$$U = \frac{EI\,\pi^4}{4\,l^3}\,[a_1^2 + 81\,a_2^2] \qquad \dots (4)$$

Work done by external force, $H = W\,y_{max}$ ... (5)

We know, Deflection, $y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$

In the span, deflection is maximum at $x = \frac{l}{2}$.

$$\Rightarrow y_{max} = a_1 \sin\frac{\pi\frac{l}{2}}{l} + a_2 \sin\frac{3\pi\frac{l}{2}}{l}$$

$$= a_1 \sin\frac{\pi}{2} + a_2 \sin\frac{3\pi}{2}$$

$$\boxed{y_{max} = a_1 - a_2} \qquad \dots (6)$$

$$[\because \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1]$$

Substitute y_{max} value in equation (5),

$$\Rightarrow \boxed{H = W(a_1 - a_2)} \qquad \dots (7)$$

Substitute U and H values in equation (2),

$$\Rightarrow \pi = \frac{EI\,\pi^4}{4\,l^3}\,[a_1^2 + 81\,a_2^2] - W(a_1 - a_2) \qquad \dots (8)$$

For stationary value of π, the following conditions must be satisfied.

$$\frac{\partial\pi}{\partial a_1} = 0 \text{ and } \frac{\partial\pi}{\partial a_2} = 0$$

$$\Rightarrow \frac{\partial\pi}{\partial a_1} = \frac{EI\,\pi^4}{4\,l^3}[2\,a_1] - W = 0$$

$$\Rightarrow \frac{EI\,\pi^4}{4\,l^3}(2\,a_1) - W = 0$$

$$\Rightarrow \frac{EI\,\pi^4}{2\,l^3}(a_1) = W$$

$$\Rightarrow \boxed{a_1 = \frac{2\,l^3\,W}{EI\,\pi^4}} \qquad \dots (9)$$

Similarly, $$\frac{\partial\pi}{\partial a_2} = \frac{EI\,\pi^4}{4\,l^3}[162\,a_2] + W = 0$$

$$\Rightarrow \frac{EI\,\pi^4}{4\,l^3}(162\,a_2) + W = 0$$

$$\Rightarrow \frac{81\,EI\,\pi^4}{2\,l^3}a_2 = -W$$

$$\Rightarrow \boxed{a_2 = \frac{-2\,l^3\,W}{81\,EI\,\pi^4}} \qquad \dots (10)$$

We know that,

Maximum deflection, $y_{max} = a_1 - a_2$

$$\Rightarrow y_{max} = \frac{2\,l^3\,W}{EI\,\pi^4} + \frac{2\,l^3\,W}{81\,EI\,\pi^4} = \frac{2\,l^3\,W}{EI\,\pi^4}\left(1 + \frac{1}{81}\right)$$

$$= \frac{2\,l^3\,W}{EI\,\pi^4}(1.0123)$$

$$= \frac{2.0246\,l^3\,W}{EI\,\pi^4} = 0.0207\,\frac{W\,l^3}{EI}$$

$$\boxed{y_{max} = \frac{W\,l^3}{48.1\,EI}} \qquad \dots (11)$$

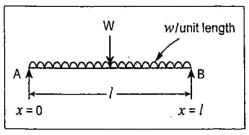
We know that, simply supported beam subjected to point load at centre, maximum deflection is,

$$y_{max} = \frac{W\,l^3}{48\,EI} \qquad \dots (12)$$

From equation (11) and (12), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are almost same. In order to get accurate result, more terms in Fourier series should be taken.

Example 1.27 ***A simply supported beam subjected to uniformly distributed load over entire span and it is subjected to a point load at the centre of the span. Calculate the bending moment and deflection at midspan by using Rayleigh-Ritz method and compare with exact solution.***

Given:



To find: 1. Deflection and Bending moment at midspan.

2. Compare with exact solutions.

☺ ***Solution:*** From Example 1.20, we know that,

$$\text{Deflection, } y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} \qquad \ldots (1)$$

Total potential energy of the beam is given by,

$$\pi = U - H \qquad \ldots (2)$$

The strain energy U of the beam due to bending is given by,

$$U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad \ldots (3)$$

From equation (9) in Example 1.18, we know that,

$$U = \frac{EI\,\pi^4}{4\,l^3}\left[a_1^2 + 81\,a_2^2\right] \qquad \ldots (4)$$

Work done by external forces,

$$H = \int_0^l \omega\, y\, dx + W\, y_{max} \qquad \ldots (5)$$

From equation (10) in Example 1.18, we know that,

$$\int_0^l \omega\, y\, dx = \frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) \qquad \ldots (6)$$

We know that, $y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$

In the span, deflection is maximum at $x = \frac{l}{2}$.

$$\Rightarrow y_{max} = a_1 \sin\frac{\pi \times \frac{l}{2}}{l} + a_2 \sin\frac{3\pi\frac{l}{2}}{l}$$

$$= a_1 \sin\frac{\pi}{2} + a_2 \sin\frac{3\pi}{2}$$

$$\boxed{y_{max} = a_1 - a_2} \qquad \ldots (7)$$

$$\left[\because \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1\right]$$

Substitute (6) and (7) values in equation (5),

$$(5) \Rightarrow \boxed{H = \frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) + W(a_1 - a_2)} \qquad \ldots (8)$$

Substituting U and H values in equation (2), we get

$$\Rightarrow \pi = \frac{EI\,\pi^4}{4\,l^3}\left[a_1^2 + 81\,a_2^2\right] - \left[\frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) + W(a_1 - a_2)\right]$$

$$\pi = \frac{EI\,\pi^4}{4\,l^3}\left[a_1^2 + 81\,a_2^2\right] - \frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) - W(a_1 - a_2) \qquad \ldots (9)$$

For stationary value of π, the following conditions must be satisfied.

$$\frac{\partial\pi}{\partial a_1} = 0 \text{ and } \frac{\partial\pi}{\partial a_2} = 0$$

$$\Rightarrow \frac{\partial\pi}{\partial a_1} = \frac{EI\,\pi^4}{4\,l^3}(2\,a_1) - \frac{2\omega l}{\pi} - W = 0$$

$$\Rightarrow \frac{EI\,\pi^4}{2\,l^3}a_1 - \frac{2\omega l}{\pi} - W = 0$$

$$\Rightarrow \frac{EI\,\pi^4}{2\,l^3}a_1 = \frac{2\omega l}{\pi} + W$$

$$\Rightarrow \boxed{a_1 = \frac{2\,l^3}{EI\,\pi^4}\left(\frac{2\omega l}{\pi} + W\right)} \qquad \ldots (10)$$

Similarly, $\frac{\partial\pi}{\partial a_2} = \frac{EI\,\pi^4}{4\,l^3}(162\,a_2) - \frac{2\omega l}{\pi}\left(\frac{1}{3}\right) + W = 0$

$$\Rightarrow \frac{EI\,\pi^4}{4\,l^3}(162\,a_2) - \frac{2\omega l}{3\pi} + W = 0$$

$$\Rightarrow \frac{EI\,\pi^4}{4\,l^3}(162\,a_2) = \frac{2\omega l}{3\pi} - W$$

$$\Rightarrow a_2 = \frac{4\,l^3}{162\,EI\,\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)$$

$$\Rightarrow \boxed{a_2 = \frac{2\,l^3}{81\,EI\,\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)} \qquad \ldots (11)$$

From equation (7), we know that,

Maximum deflection, $y_{max} = a_1 - a_2$

$$\Rightarrow y_{max} = \frac{2\,l^3}{EI\,\pi^4}\left(\frac{2\omega l}{\pi} + W\right) - \frac{2\,l^3}{81\,EI\,\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)$$

$$\Rightarrow y_{max} = \frac{4\omega l^4}{EI\,\pi^5} + \frac{2W\,l^3}{EI\,\pi^4} - \frac{4\omega\,l^4}{243\,EI\,\pi^5} + \frac{2W\,l^3}{81\,EI\,\pi^4}$$

$$= \frac{4\omega l^4}{EI\,\pi^5}\left(1 - \frac{1}{243}\right) + \frac{2\,W\,l^3}{EI\,\pi^4}\left(1 + \frac{1}{81}\right)$$

$$= \frac{3.98\,\omega\,l^4}{EI\,\pi^5} + \frac{2.02\,W\,l^3}{EI\,\pi^4}$$

$$= \left[0.0130\,\frac{\omega\,l^4}{EI} + 0.0207\,\frac{W\,l^3}{EI}\right]$$

$$\boxed{y_{max} = \left[0.0130\,\frac{\omega\,l^4}{EI} + 0.0207\,\frac{W\,l^3}{EI}\right]} \quad \ldots (12)$$

We know that, simply supported beam subjected to uniformly distributed load, maximum deflection is,

$$y_{max} = \frac{5}{384}\,\frac{\omega\,l^4}{EI}$$

Simply supported beam subjected to point load at centre, maximum deflection is,

$$y_{max} = \frac{W\,l^3}{48\,EI}$$

So, Total deflection, $y_{max} = \frac{5}{384}\,\frac{\omega\,l^4}{EI} + \frac{W\,l^3}{48\,EI}$

$$\boxed{y_{max} = 0.0130\,\frac{\omega\,l^4}{EI} + 0.0208\,\frac{W\,l^3}{EI}} \quad \ldots (13)$$

From equations (12) and (13), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are same.

Bending Moment at Midspan

We know that,

$$\text{Bending moment, } M = EI\,\frac{d^2y}{dx^2} \quad \ldots (14)$$

From equation (4), we know that,

$$\frac{d^2y}{dx^2} = -\left[\frac{a_1\,\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{a_2\,9\,\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]$$

Substitute a_1 and a_2 values from equation (10) and (11),

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{2\,l^3}{EI\,\pi^4}\left(\frac{2\omega l}{\pi} + W\right)\times\frac{\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{2\,l^3}{81\,EI\,\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)\times\frac{9\pi^2}{l^2}\times\sin\frac{3\pi x}{l}\right]$$

Maximum bending occurs at $x = \frac{l}{2}$

$$= -\left[\frac{2\,l^3}{EI\,\pi^4}\left(\frac{2\omega l}{\pi} + W\right)\times\frac{\pi^2}{l^2}\sin\frac{\pi\times\frac{l}{2}}{l} + \frac{2\,l^3}{81\,EI\,\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)\times\frac{9\pi^2}{l^2}\times\sin\frac{3\pi\,\frac{l}{2}}{l}\right]$$

$$= -\left[\frac{2\,l^3}{EI\,\pi^4}\left(\frac{2\omega l}{\pi} + W\right)\times\frac{\pi^2}{l^2}(1) + \frac{2\,l^3}{81\,EI\,\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)\times\frac{9\pi^2}{l^2}(-1)\right]$$

$$\left[\because \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1\right]$$

$$= -\left[\frac{2\,l}{EI\,\pi^2}\left(\frac{2\omega l}{\pi} + W\right) - \frac{2\,l}{9\pi^2\,EI}\left(\frac{2\omega l}{3\pi} - W\right)\right]$$

$$= -\left[\frac{4\,\omega\,l^2}{EI\,\pi^3} + \frac{2W\,l}{EI\pi^2} - \frac{4\,\omega\,l^2}{27\,\pi^3\,EI} + \frac{2\,W\,l}{9\,\pi^2\,EI}\right]$$

$$= -\left[\frac{4\,\omega\,l^2}{EI\,\pi^3} - \frac{4\,\omega\,l^2}{27\,\pi^3\,EI} + \frac{2W\,l}{EI\pi^2} + \frac{2\,W\,l}{9\,\pi^2\,EI}\right]$$

$$= -\left[\frac{4\,\omega\,l^2}{EI\,\pi^3}\left(1 - \frac{1}{27}\right) + \frac{2W\,l}{EI\,\pi^2}\left(1 + \frac{1}{9}\right)\right]$$

$$= -\left[\frac{3.8518\,\omega\,l^2}{EI\,\pi^3} + 2.222\,\frac{W\,l}{EI\,\pi^2}\right]$$

$$\boxed{\frac{d^2y}{dx^2} = -\left[0.124\,\frac{\omega\,l^2}{EI} + 0.225\,\frac{W\,l}{EI}\right]}$$

Substitute $\frac{d^2y}{dx^2}$ value in bending moment equation,

$$(14) \Rightarrow \quad M_{centre} = EI\,\frac{d^2y}{dx^2} = -EI\left[0.124\,\frac{\omega\,l^2}{EI} + 0.225\,\frac{W\,l}{EI}\right]$$

$$\boxed{M_{centre} = -(0.124\,\omega\,l^2 + 0.225\,W\,l)} \quad \ldots (15)$$

[*Note: Negative sign indicates downward deflection*]

We know that, simply supported beam subjected to uniformly distributed load, maximum bending moment is,

$$M_{centre} = \frac{\omega l^2}{8}$$

Simply supported beam subjected to point load at centre, maximum bending moment is,

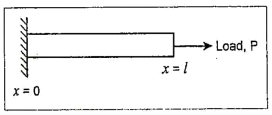
$$M_{centre} = \frac{W l}{4}$$

Total bending moment, $M_{centre} = \frac{\omega l^2}{8} + \frac{W l}{4}$

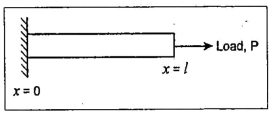
$$M_{centre} = 0.125\,\omega l^2 + 0.25\,W l \qquad \ldots (16)$$

From equation (15) and (16), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are almost same. In order to get accurate results, more terms in Fourier series should be taken.

Example 1.28 ***A bar of uniform cross section is clamped at one end and left free at the other end and it is subjected to a uniform axial load P as shown in Fig. Calculate the displacement and stress in a bar by using two terms polynomial and three terms polynomial. Compare with exact solutions.***



Given:



To find: 1. Displacement of the bar, δu.

2. Stress in the bar, σ.

By using two terms and three terms polynomial.

☺ ***Solution :*** We know that, Polynomial function for displacement is,

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\ldots + a_n x^n$$

Case (i): Considering two terms of polynomial,

$$i.e.,\ u = a_0 + a_1 x \qquad \ldots (1)$$

Apply boundary condition,

at $x = 0$, $u = 0$

$$\Rightarrow \quad 0 = a_0 + 0$$

$$\Rightarrow \quad \boxed{a_0 = 0}$$

substituting a_0 value in equation (1),

$$\Rightarrow \quad u = a_1 x \qquad \ldots (2)$$

$$\boxed{\frac{du}{dx} = a_1}$$

We know that,

Total potential energy of the bar, $\pi = U - H$ ... (3)

where, U → Strain energy of the bar.

H → Work done by external force of the bar.

$$\text{Strain energy, } U = \frac{EA}{2}\int_0^l \left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{EA}{2}\int_0^l (a_1)^2\, dx$$

$$= \frac{EA\, a_1^2}{2}\,[x]_0^l$$

$$\boxed{U = \frac{EA\, a_1^2\, l}{2}} \qquad \ldots (4)$$

$$\text{Work done by external force, } H = \int_0^l P\, dx = \int_0^l \rho\, u\, A\, dx \qquad [\because \text{Load, } P = \rho\, u\, A]$$

$$= \rho A \int_0^l u\, dx = \rho A \int_0^l a_1 x\, dx \qquad [\because u = a_1 x]$$

$$= \rho A\, a_1 \left[\frac{x^2}{2}\right]_0^l = \frac{\rho A\, a_1}{2}[l^2]$$

$$\boxed{H = \frac{\rho A\, a_1\, l^2}{2}} \qquad \ldots (5)$$

Substitute U and H values in equation (3),

$$(3) \Rightarrow \qquad \pi = \frac{E A a_1^2 l}{2} - \frac{\rho A a_1 l^2}{2}$$

For stationary value of π, the following condition must be satisfied.

$$i.e., \quad \frac{\partial \pi}{\partial a_1} = 0$$

$$\Rightarrow \quad \frac{E A (2 a_1) l}{2} - \frac{\rho A l^2}{2} = 0$$

$$\Rightarrow \quad E A a_1 l - \frac{\rho A l^2}{2} = 0$$

$$\Rightarrow \quad E A a_1 l = \frac{\rho A l^2}{2}$$

$$\Rightarrow \quad \boxed{a_1 = \frac{\rho l}{2 E}}$$

Substitute a_1 value in equation (2),

$$\Rightarrow \quad u = a_1 x = \frac{\rho l}{2 E} \times x$$

$$\boxed{u = \frac{\rho l}{2E} \times x}$$

We know that, Extension of the bar, $\delta u = u_1 - u_0 = \frac{\rho l}{2E} \times l - 0$

$$[\because \text{At } x = l,\ u = u_1, \quad \text{At } x = 0, u = u_0 = 0]$$

$$= \frac{\rho l^2}{2E}$$

$$\boxed{\text{Extension or displacement of the bar, } \delta u = \frac{\rho l^2}{2E}} \qquad \ldots (6)$$

$$\text{Stress in the bar, } \sigma = E \frac{du}{dx} = E \times \frac{\rho l}{2E} \qquad \left[\because u = \frac{\rho l}{2E} \times x\right]$$

$$\boxed{\text{Stress in the bar, } \sigma = \frac{\rho l}{2}} \qquad \ldots (7)$$

Case (ii): Considering three terms of polynomial,

$$i.e., \quad u = a_0 + a_1 x + a_2 x^2 \qquad \ldots (8)$$

Apply boundary condition, at $x = 0$, $u = 0$

$$\Rightarrow \quad 0 = a_0 + 0 + 0$$

$$\Rightarrow \boxed{a_0 = 0}$$

Substitute a_0 value in equation (8),

$$(8) \Rightarrow \quad u = a_1 x + a_2 x^2 \qquad \ldots (9)$$

$$\boxed{\frac{du}{dx} = a_1 + 2 a_2 x}$$

We know that,

Total potential energy of the bar, $\pi = U - H$ ... (10)

$$\text{Strain energy, } U = \frac{EA}{2} \int_0^l \left(\frac{du}{dx}\right)^2 dx = \frac{EA}{2} \int_0^l (a_1 + 2 a_2 x)^2 dx$$

$$= \frac{EA}{2} \int_0^l [a_1^2 + (2 a_2 x)^2 + 2 a_1 2 a_2 x] dx$$

$$[\because (a+b)^2 = a^2 + b^2 + 2ab]$$

$$= \frac{EA}{2} \int_0^l [a_1^2 dx + 4 a_2^2 x^2 dx + 4 a_1 a_2 x dx]$$

$$= \frac{EA}{2} \left[a_1^2 (x)_0^l + 4 a_2^2 \left(\frac{x^3}{3}\right)_0^l + 4 a_1 a_2 \left(\frac{x^2}{2}\right)_0^l \right]$$

$$= \frac{EA}{2} \left[a_1^2 (l - 0) + \frac{4 a_2^2}{3} (l^3 - 0) + \frac{4 a_1 a_2}{2} (l^2 - 0) \right]$$

$$U = \frac{EA}{2} \left[a_1^2 l + \frac{4 a_2^2}{3} (l^3) + 2 a_1 a_2 (l^2) \right] \qquad \ldots (11)$$

Work done by external force,

$$H = \int_0^l P\, dx = \int_0^l \rho u A\, dx \qquad [\because \text{Load, } P = \rho u A]$$

$$= \rho A \int_0^l u\, dx = \rho A \int_0^l (a_1 x + a_2 x^2) dx$$

$$= \rho A \int_0^l [a_1 x\, dx + a_2 x^2 dx]$$

$$= \rho A\left[a_1\left(\frac{x^2}{2}\right)_0^l + a_2\left(\frac{x^3}{3}\right)_0^l\right]$$

$$= \rho A\left[\frac{a_1}{2}(l^2-0) + \frac{a_2}{3}(l^3-0)\right]$$

$$H = \rho A\left[\frac{a_1}{2}l^2 + \frac{a_2}{3}l^3\right] \quad \text{... (12)}$$

Substitute (11) and (12) values in (10),

(10) $\Rightarrow$ $\pi = U - H$

$$\pi = \frac{EA}{2}\left[a_1^2 l + \frac{4a_2^2}{3}(l^3) + 2a_1a_2(l^2)\right] - \rho A\left[\frac{a_1}{2}l^2 + \frac{a_2}{3}l^3\right] \quad \text{... (13)}$$

For stationary value of π, the following conditions must be satisfied.

i.e., $\frac{\partial\pi}{\partial a_1} = 0$ and $\frac{\partial\pi}{\partial a_2} = 0$

$$\Rightarrow \frac{\partial\pi}{\partial a_1} = \frac{EA}{2}\left[2a_1 l + 0 + 2a_2 l^2\right] - \rho A\left[\frac{l^2}{2} + 0\right] = 0$$

$$\Rightarrow \frac{EA}{2}[2a_1 l + 2a_2 l^2] - \rho A\left[\frac{l^2}{2}\right] = 0$$

$$\Rightarrow EA[a_1 l + a_2 l^2] - \frac{\rho A l^2}{2} = 0$$

$$\Rightarrow EA[a_1 l + a_2 l^2] = \frac{\rho A l^2}{2}$$

$$\Rightarrow a_1 l + a_2 l^2 = \frac{\rho l^2}{2E}$$

$$\Rightarrow a_1 + a_2 l = \frac{\rho l}{2E} \quad \text{... (14)}$$

Similarly, $\frac{\partial\pi}{\partial a_2} = 0$

$$\Rightarrow \frac{EA}{2}\left[0 + \frac{8a_2}{3}l^3 + 2a_1 l^2\right] - \rho A\left[0 + \frac{l^3}{3}\right] = 0$$

$$\Rightarrow \frac{EA}{2}\left[\frac{8}{3}a_2 l^3 + 2a_1 l^2\right] = \frac{\rho A l^3}{3}$$

$$\Rightarrow \frac{8}{3}a_2 l^3 + 2a_1 l^2 = \frac{2\rho A l^3}{3EA}$$

$$\Rightarrow \frac{8}{3}a_2 l^3 + 2a_1 l^2 = \frac{2\rho l^3}{3E}$$

$$\Rightarrow \frac{4}{3}a_2 l^3 + a_1 l^2 = \frac{\rho l^3}{3E}$$

$$\Rightarrow \frac{4}{3}a_2 l + a_1 = \frac{\rho l}{3E}$$

$$\Rightarrow a_1 + \frac{4}{3}a_2 l = \frac{\rho l}{3E}$$

$$\Rightarrow \boxed{a_1 + 1.333\,a_2 l = \frac{\rho l}{3E}} \quad \text{... (15)}$$

Solving (14) and (15),

$$a_1 + a_2 l = \frac{\rho l}{2E}$$

$$a_1 + 1.333\,a_2 l = \frac{\rho l}{3E}$$

$$a_2 l - 1.333\,a_2 l = \frac{\rho l}{2E} - \frac{\rho l}{3E}$$

$$-0.333\,a_2 l = \frac{\rho l}{E}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\rho l}{E}\left(\frac{3-2}{6}\right)$$

$$-0.333\,a_2 l = \frac{\rho l}{6E}$$

$$\Rightarrow -0.333\,a_2 = \frac{\rho}{6E}$$

$$\Rightarrow \boxed{a_2 = \frac{-\rho}{2E}}$$

Substituting a_2 value in equation (14),

(14) $\Rightarrow$ $a_1 + \left(\frac{-\rho}{2E}l\right) = \frac{\rho l}{2E}$

$$a_1 - \frac{\rho l}{2E} = \frac{\rho l}{2E}$$

$$a_1 = \frac{\rho l}{2E} + \frac{\rho l}{2E}$$

$$a_1 = \frac{2\rho l}{2E}$$

$$\boxed{a_1 = \frac{\rho l}{E}}$$

We know that, $u = a_1 x + a_2 x^2$

Substitute a_1 and a_2 values, $u = \frac{\rho l}{E}x - \frac{\rho}{2E}x^2$

$$\boxed{u = \frac{\rho}{E}\left[lx - \frac{x^2}{2}\right]} \quad \ldots (16)$$

At $x = l$, $u = u_1$ substitute in equation (16),

$$\Rightarrow u_1 = \frac{\rho}{E}\left[l^2 - \frac{l^2}{2}\right]$$

$$\boxed{u_1 = \frac{\rho}{E} \times \frac{l^2}{2}}$$

We know that, Extension of the bar, $\delta u = u_1 - u_0 = \frac{\rho l^2}{2E} - 0$ [$\because$ At $x = 0$, $u = u_0 = 0$]

$$= \frac{\rho l^2}{2E}$$

$$\boxed{\text{Extension or displacement of the bar} = \frac{\rho l^2}{2E}} \quad \ldots (17)$$

From equation (16), we know that,

$$u = \frac{\rho}{E}\left(lx - \frac{x^2}{2}\right)$$

$$\frac{du}{dx} = \frac{\rho}{E}\left(l - \frac{2x}{2}\right) = \frac{\rho}{E}(l - x)$$

$$\text{Stress in the bar, } \sigma = E\frac{du}{dx} = E \times \frac{\rho}{E}(l - x) = \rho(l - x)$$

$$\boxed{\text{Stress in the bar, } \sigma = \rho(l - x)} \quad \ldots (18)$$

Exact solution: We know that, actual extension of the bar,

$$\delta l = \int_0^l \frac{P\,dx}{AE} = \int_0^l \frac{\rho A x}{AE}\,dx \qquad [\because P = \rho A x]$$

$$= \frac{\rho}{E}\int_0^l x\,dx = \frac{\rho}{E}\left[\frac{x^2}{2}\right]_0^l = \frac{\rho}{E}\left[\frac{l^2}{2}\right]$$

$$\boxed{\delta l = \frac{\rho l^2}{2E}} \quad \ldots (19)$$

From equation (6), (17) and (19), we know that total extension of the bar obtained is exact in both the cases.

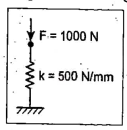
Result: 1. Displacement of the bar, $\delta u = \frac{\rho l^2}{2E}$ [Two terms polynomial]

$\delta u = \frac{\rho l^2}{2E}$ [Three terms polynomial]

2. Stress in the bar, $\sigma = \frac{\rho l}{2}$ [Two terms polynomial]

$\sigma = \rho(l - x)$ [Three terms polynomial]

Example 1.29 ***A linear elastic spring is subjected to a force of 1000 N as shown in Fig. Calculate the displacement and the potential energy of the spring.***



Given: Force, F = 1000 N

Stiffness, k = 500 N/mm

To find: 1. Displacement, x.

2. Potential energy, π.

☺ ***Solution :*** We know that,

$$\text{Total potential energy, } \pi = U - H \quad \ldots (1)$$

$$\text{where, } U = \text{Strain energy} = \frac{1}{2}(kx) \times x$$

$$H = \text{Work done by external force} = Fx$$

Substitute U and H values in equation (1),

$$\Rightarrow \pi = \frac{1}{2}(kx) \times x - Fx$$

$$\pi = \frac{1}{2}kx^2 - Fx \quad \ldots (2)$$

For stationary value of π, $\frac{\partial \pi}{\partial x} = 0$

$$\Rightarrow \frac{1}{2} \times 2kx - F = 0$$

$$\Rightarrow kx - F = 0$$

$$\Rightarrow 500(x) - 1000 = 0$$

$$\Rightarrow \quad 500\,x = 1000$$

$$\Rightarrow \quad \boxed{x = 2 \text{ mm}}$$

Substitute x value in equation (2),

$$(2) \Rightarrow \quad \pi = \frac{1}{2} k x^2 - F x = \frac{1}{2}(500)(2)^2 - 1000\,(2)$$

$$\boxed{\pi = -1000 \text{ N-mm}}$$

Result: 1. Displacement, $x = 2$ mm

2. Potential energy, $\pi = -1000$ N-mm

Example 1.30 ***Consider a 1 mm diameter, 50 mm long aluminium pin-fin as shown in Fig.(i) used to enhance the heat transfer from a surface wall maintained at 300° C. Calculate the temperature distribution in a pinfin by using Rayleigh-Ritz method. Take, k = 200 w/m° C for aluminium h = 20 w/m² °C, T_∞ = 30° C.***

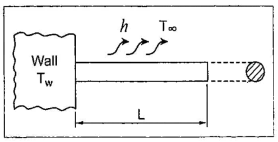


Fig. (i).

$$k\frac{d^2T}{dx^2} = \frac{P\,h}{A}(T - T_\infty)$$

$$T(0) = T_w = 300°C$$

$$q_L = kA\frac{dT}{dx}(L) = 0 \text{ (Insulated tip)}$$

[Anna University, M.E. (CAD/CAM) Apr/May 2006]

Given: The governing differential equation,

$$k\frac{d^2T}{dx^2} = \frac{P\,h}{A}(T - T_\infty)$$

Diameter, $d = 1$ mm $= 1 \times 10^{-3}$ m

Length, $l = 50$ mm $= 50 \times 10^{-3}$ m

Thermal conductivity, $k = 200$ W/m °C

Heat transfer coefficient, $h = 20$ W/m² °C

Fluid temperature, $T_\infty = 30°C$

Boundary conditions, $T(0) = T_w = 300°C$

$$q_L = k\,A\frac{dT}{dx}(L) = 0$$

To find: Ritz parameters.

☺ ***Solution:*** The equivalent functional representation is given by

$$\pi = \text{Strain energy} - \text{Work done}$$

$$\pi = U - W$$

$$\pi = \left[\int_0^L \frac{1}{2} k\left(\frac{dT}{dx}\right)^2 dx + \int_0^L \frac{1}{2}\frac{P\,h}{A}(T - T_\infty)^2\,dx\right] - q_L\,T_L \quad \ldots(1)$$

$$\pi = \int_0^L \frac{1}{2} k\left(\frac{dT}{dx}\right)^2 dx + \int_0^L \frac{1}{2}\frac{P\,h}{A}(T - T_\infty)^2\,dx \quad \ldots(2)$$

$[\because q_L = 0]$

Assume a trial function, let $T(x) = a_0 + a_1 x + a_2 x^2$ ...(3)

Apply boundary condition,

at $x = 0$, $T(x) = 300$

$$300 = a_0 + a_1(0) + a_2(0)^2$$

$$\boxed{a_0 = 300}$$

Substituting a_0 value in equation (3),

$$T(x) = 300 + a_1 x + a_2 x^2 \quad \ldots(4)$$

$$\Rightarrow \frac{dT}{dx} = a_1 + 2\,a_2\,x \quad \ldots(5)$$

Substitute the equation (4), (5) in (2)

$$\pi = \int_0^l \frac{1}{2} k(a_1 + 2\,a_2\,x)^2\,dx + \int_0^l \frac{1}{2}\frac{P\,h}{A}(300 + a_1 x + a_2 x^2 - 30)^2\,dx$$

$$\pi = \int_0^l \frac{1}{2} k (a_1 + 2 a_2 x)^2 dx + \int_0^l \frac{1}{2} \frac{P h}{A} (270 + a_1 x + a_2 x^2)^2 dx$$

$$[\because (a+b)^2 = a^2 + b^2 + 2ab; \ (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca]$$

$$\pi = \frac{k}{2} \int_0^l [a_1^2 + 4 a_2^2 x^2 + 4 a_1 a_2 x] +$$

$$\frac{P h}{2A} \int_0^l [(270)^2 + a_1^2 x_1^2 + a_2^2 x^4 + 540 a_1 x + 2 a_1 a_2 x^3 + 540 a_2 x^2)] dx$$

$$\pi = \frac{k}{2} \left[a_1^2 x + \frac{4 a_2^2 x^3}{3} + \frac{4 a_1 a_2 x^2}{2} \right]_0^{50 \times 10^{-3}}$$

$$+ \frac{P h}{2A} \left[72900 x + \frac{a_1^2 x^3}{3} + \frac{a_2^2 x^5}{5} + \frac{540 a_1 x^2}{2} + \frac{2 a_1 a_2 x^4}{4} + \frac{540 a_2 x^3}{3} \right]_0^{50 \times 10^{-3}}$$

$$[\because l = 50 \times 10^{-3} \text{ m}]$$

$$\pi = \frac{k}{2} \left[(50 \times 10^{-3}) a_1^2 + \frac{4 a_2^2 (50 \times 10^{-3})^3}{3} + \frac{4 a_1 a_2 (50 \times 10^{-3})^2}{2} \right]$$

$$+ \frac{P h}{2A} \left[72900 (50 \times 10^{-3}) + \frac{a_1^2 (50 \times 10^{-3})^3}{3} + \frac{a_2^2 (50 \times 10^{-3})^5}{5} \right.$$

$$\left. + \frac{540 a_1 (50 \times 10^{-3})^2}{2} + \frac{2 a_1 a_2 (50 \times 10^{-3})^4}{4} + \frac{540 a_2 (50 \times 10^{-3})^3}{3} \right]$$

$$\pi = \frac{200}{2} [50 \times 10^{-3} a_1^2 + 1.666 \times 10^{-4} a_2^2 + 5 \times 10^{-3} a_1 a_2] + \frac{\pi \times 10^{-3} \times 20}{2 \times \frac{\pi}{4} \times (10^{-3})^2}$$

$$[3645 + 4.166 \times 10^{-5} a_1^2 + 6.25 \times 10^{-8} a_2^2 + 0.675 a_1 + 3.125 \times 10^{-6} a_1 a_2 + 0.0225 a_2]$$

$$\pi = [5 a_1^2 + 0.0166 a_2^2 + 0.5 a_1 a_2] + [14.58 \times 10^7 + 1.66 a_1^2 + 2.5 \times 10^{-3} a_2^2$$

$$+ 2700 a_1 + 0.125 a_1 a_2 + 900 a_2]$$

$$\pi = [6.66 a_1^2 + 0.0191 a_2^2 + 0.625 a_1 a_2 + 27000 a_1 + 900 a_2 + 14.58 \times 10^7]$$

Apply, $\frac{\partial \pi}{\partial a_1} = 0$

$\Rightarrow \quad 13.32 a_1 + 0.625 a_2 + 27000 = 0$

$\Rightarrow \quad 13.32 a_1 + 0.625 a_2 = -27000 \quad ...(6)$

Apply, $\frac{\partial \pi}{\partial a_2} = 0$

$\Rightarrow \quad 0.625 a_1 + 0.382 a_2 + 900 = 0$

$\Rightarrow \quad 0.625 a_1 + 0.0382 a_2 = -900 \quad ...(7)$

Solve the equations (6) and (7)

$$13.32 a_1 + 0.625 a_2 = -27000 \quad ...(6)$$

$$0.625 a_1 + 0.0382 a_2 = -900 \quad ...(7)$$

(6) × 0.625

$\Rightarrow \quad 8.325 a_1 + 0.3906 a_2 = -16875 \quad ...(8)$

(7) × – 13.32

$\Rightarrow \quad -8.325 a_1 - 0.5088 a_2 = +11988 \quad ...(9)$

$$-0.1182 a_2 = -4887$$

$$0.1182 a_2 = 4887$$

$$\boxed{a_2 = 41345}$$

Substitute a_2 value in equation (6),

$$13.32 a_1 + 0.625 (41345) = -27000$$

$$\Rightarrow a_1 = -3967.01$$

Substitute a_0, a_1 and a_2 values in equation (3)

$$T = 300 - 3967.01 x + 41345 x^2$$

Result: Temperature distribution in a pin-fin

$$T = 300 - 3967.01 x + 41345 x^2$$

Example 1.31 ***Using Rayleigh-Ritz method, determine the expressions for displacement and stress in a fixed bar subjected to axial force P as shown in Fig.(i). Draw the displacement and stress variation diagram. Take three terms in displacement function.***

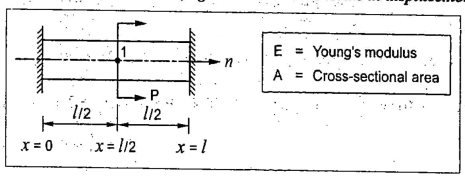


Fig. (i).

Given:

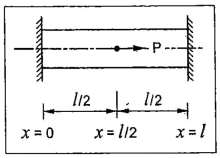


Fig. (ii).

To draw: Displacement and stress variation diagram

☺ ***Solution:*** We know that, polynomial function for three terms,

$$i.e., \quad u = a_0 + a_1 x + a_2 x^2 \qquad ...(1)$$

This function has to satisfy the boundary conditions,

(i) at $x = 0$, $u = 0$

(ii) at $x = l$, $u = 0$

Apply boundary condition (i), we get

$$\boxed{a_0 = 0} \qquad ...(2)$$

Apply boundary condition (ii), we get

$$0 = a_0 + a_1 l + a_2 l^2 \qquad ...(3)$$

∴ From equations (2) and (3) we get,

$$0 = a_1 l + a_2 l^2$$

$$\boxed{a_1 = -a_2 l} \qquad ...(4)$$

Substituting a_0 and a_1 values in equation (1),

$$u = 0 - a_2 l x + a_2 x^2$$

$$\boxed{u = a_2 [-l x + x^2]} \qquad ...(5)$$

At $x = \frac{l}{2}$,

$$u_1 = a_2 \left[-l \left(\frac{l}{2}\right) + \left(\frac{l}{2}\right)^2 \right]$$

$$\boxed{u_1 = -\frac{a_2 l^2}{4}} \qquad ...(6)$$

We know that, Total potential energy of the bar,

$$\pi = U - H$$

Where, U → Strain energy of the bar,

H → Work done by external force of the bar.

∴ Potential energy

$$\pi = \frac{EA}{2} \int_0^l \left(\frac{du}{dx}\right)^2 dx - p\, u_1 \qquad ...(7)$$

We know that, $u = a_2 (-l x + x^2)$

$$\Rightarrow \quad \boxed{\frac{du}{dx} = a_2 (-l + 2x)}$$

Now,

$$\pi = \frac{EA}{2} \int_0^l [a_2 (-l + 2x)]^2 dx - p\left(\frac{-a_2 l^2}{4}\right)$$

$$= \frac{EA}{2} a_2^2 \int_0^l (l^2 + 4x^2 - 4 l x)\, dx + p\, a_2 \frac{l^2}{4}$$

$$= \frac{EA}{2} a_2^2 \left[l^2 x + \frac{4x^3}{3} - 2 l x^2 \right]_0^l + p\, a_2 \frac{l^2}{4}$$

$$= \frac{EA}{2} a_2^2 \left[l^3 + \frac{4 l^3}{3} - 2 l (l^2) \right] + p\, a_2 \frac{l^2}{4}$$

$$\pi = \frac{EA}{2} a_2^2 \left(\frac{l^3}{3}\right) + p\, a_2 \frac{l^2}{4}$$

For stationary value of π, the following conditions must be satisfied,

i.e.,

$$\frac{\partial \pi}{\partial a_2} = 0$$

$$\frac{EA}{2}\left(2 a_2 \frac{l^3}{3}\right) + p \frac{l^2}{4} = 0$$

$$EA\, a_2 \frac{l^3}{3} = p \frac{l^2}{4}$$

$$\boxed{a_2 = \frac{-3p}{4EAl}} \qquad ...(8)$$

Substitute a_2 value in equations (5) and (6)

∴

$$\boxed{u = \frac{-3p}{4EA\,l}[-l x + x^2]}$$

$$\boxed{u_1 = \frac{3 p l}{16}}$$

$\therefore$ Stress in the bar, $\sigma = E\dfrac{du}{dx}$

$$= E\,a_2\,(-l + 2x)$$

$$= \frac{-3\,E\,p}{4\,E\,A\,l}\,[-l + 2x]$$

$$\boxed{\sigma = \frac{3p}{4A\,l}\,[l - 2x]}$$

We know that,

At $x = 0 \Rightarrow$ $\quad \sigma_0 = \sigma_{x=0} = \dfrac{3p}{4A}$

At $x = \dfrac{l}{2} \Rightarrow$ $\quad \sigma_1 = \sigma_{x=\frac{l}{2}} = 0$

At $x = l \Rightarrow$ $\quad \sigma_2 = \sigma_{x=l} = -\dfrac{3p}{4A}$

The variation of displacement and stress diagram are shown in figure.

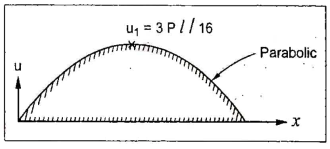


Fig. Variation of displacement

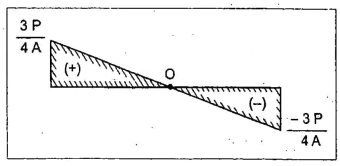


Fig. Variation of stress

Example 1.32 ***Consider the differential equation for a problem such as $\dfrac{d^2y}{dx^2} + 300\,x^2 = 0;\ 0 \le x \le 1$ with the boundary conditions, $y(0) = y(1) = 0$, the functional corresponding to this problem to be extremized is given by***

$$I = \int_0^l \left\{ -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 300\,x^2\,y \right\} dx$$

Find the solution of the problem using Rayleigh Ritz method using a one term solution is $y = a\,x\,(1 - x^3)$.

Given: Differential equation,

$$\frac{d^2y}{dx^2} + 300\,x^2 = 0; \qquad 0 \le x \le 1$$

Boundary conditions: $\quad y(0) = y(1) = 0$

$(i)\ x = 0,\ y = 0$

$(ii)\ x = 1,\ y = 0$

$$I = \int_0^l \left\{ -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 300\,x^2\,y \right\} dx$$

Trial function, $\quad y = a\,x\,(1 - x^3)$

To find: Solution of the problem by using Rayleigh – Ritz method.

© ***Solution:*** The trial function is,

$$y = a\,x\,(1 - x^3)$$

$$i.e.,\ y = a\,x - a\,x^4 \qquad \ldots(1)$$

It satisfies the boundary conditions,

$$x = 0, \quad y = 0$$

$$x = 1, \quad y = 0$$

Differential equation,

$$\frac{d^2y}{dx^2} + 300\,x^2 = 0 \qquad \ldots(2)$$

$$\Rightarrow \frac{dy}{dx} = a - 4\,a\,x^3$$

$$\left(\frac{dy}{dx}\right)^2 = (a - 4\,a\,x^3)^2 \qquad \ldots(3)$$

We know that, $$I = \int_0^l \left\{ -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 300\,x^2\,y \right\} dx \qquad \ldots(4)$$

Substitute the equation (1), (3) in (4)

$$\Rightarrow \quad I = \int_0^1 \left[-\frac{1}{2}(a - 4\,a\,x^3)^2 + 300\,x^2\,(a\,x - a\,x^4) \right] dx$$

$$= \int_0^1 \left[-\frac{1}{2}\,(a^2 + 16\,a^2\,x^6 - 8\,a^2\,x^3) + (300\,a\,x^3 - 300\,a\,x^6) \right] dx$$

$$= -\frac{1}{2}\left[a^2\,x + 16\,a^2\frac{x^7}{7} - 8\,a_2\frac{x^4}{4} \right]_0^1 + \left[300\frac{a\,x^4}{4} - 300\frac{a\,x^7}{7} \right]_0^1$$

$$= -\frac{1}{2}\left[a^2 + \frac{16}{7}\,a^2 - 2\,a^2 \right] + \frac{300}{4}\,a - \frac{300}{7}\,a$$

$$I = -\frac{a^2}{2} - \frac{8}{7}\,a^2 + a^2 + \frac{300}{4}\,a - \frac{300}{7}\,a$$

Apply, $\dfrac{\partial I}{\partial a} = 0$

$$\Rightarrow \quad \frac{-2\,a}{2} - \frac{8}{7}\,(2\,a) + 2\,a + \frac{300}{4} - \frac{300}{7} = 0$$

$$-a - \frac{16}{7}\,a + 2\,a + \frac{300}{4} - \frac{300}{7} = 0$$

$$i.e., \quad \frac{-16\,a + 7\,a}{7} = \frac{-2100 + 1200}{28}$$

$$i.e., \quad -9\,a = -900 \times \frac{7}{28}$$

$$\boxed{a = 25}$$

Hence the solution is, $\boxed{y = 25\,x(1 - x^3)}$

Result: Solution, $y = 25\,x(1 - x^3)$.

1.21. APPLICATION TO BAR ELEMENT

1.21.1. Bar Element Formulated from the Stationarity of a Functional

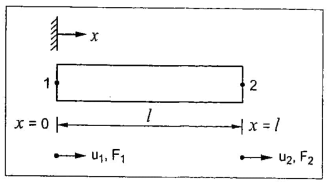


Fig. 1.31. Typical bar Element

Consider a bar element with nodes 1 and 2 as shown in Fig.1.31. u_1 and u_2 are the displacements at the respective nodes. So, u_1 and u_2 are considered as degrees of freedom of this bar element. [Refer Chapter 2, Section 2.6.3, equation no.(2.21)].

[***Note:*** Degrees of freedom is nothing but noded displacements.]

$$u = N_1\,u_1 + N_2\,u_2 \qquad \ldots(1.27)$$

$$\text{Where,} \quad N_1 = 1 - \frac{x}{l}$$

$$N_2 = \frac{x}{l}$$

Substitute the N_1, N_2 values in equation (1.27)

$$u = \left(1 - \frac{x}{l}\right)u_1 + \left(\frac{x}{l}\right)u_2 \qquad \ldots(1.28)$$

The strain energy stored within the element is given by,

$$u = \int_0^l \frac{A\,E}{2}\left(\frac{du}{dx}\right)^2 dx \qquad \ldots(1.29)$$

$$u = \frac{A\,E}{2}\left(\frac{u_2 - u_1}{l}\right)^2 \int_0^l dx$$

$$u = \frac{A\,E}{2}\left(\frac{u_2 - u_1}{l}\right)^2 (x)_0^l$$

$$\boxed{u = \frac{A\,E}{2}\,\frac{(u_2 - u_1)^2}{l}\,(l)} \qquad \ldots(1.30)$$

When there is a distributed force q_0 acting at each point on the element and concentrated forces F at the nodes, the potential of the external forces is given by,

$$H = \int_0^l q_0 u\, dx + F_1 u_1 + F_2 u_2$$

$$= q_0 \left(\frac{u_1 + u_2}{2}\right) l + F_1 u_1 + F_2 u_2 \qquad \left[\because u = \frac{u_1 + u_2}{2}\right]$$

$$H = q_0 \frac{l}{2} (u_1 + u_2) + F_1 u_1 + F_2 u_2 \qquad \text{... (1.31)}$$

Thus the total potential energy,

$$\pi = U - H$$

$$\pi = \frac{AE}{2} \frac{(u_2 - u_1)^2}{l} - q_0 \frac{l}{2} (u_1 + u_2) - F_1 u_1 - F_2 u_2 \qquad \text{... (1.32)}$$

Apply, $\dfrac{\partial \pi}{\partial u_1} = 0$

$$\Rightarrow -\frac{AE}{2l} \times 2 (u_2 - u_1) - \frac{q_0 l}{2} - F_1 = 0$$

$$\frac{AE}{l} (u_1 - u_2) - \frac{q_0 l}{2} - F_1 = 0$$

$$\Rightarrow \frac{AE}{l} (u_1 - u_2) = \frac{q_0 l}{2} + F_1 \qquad \text{... (1.33)}$$

Similarly, $\dfrac{\partial \pi}{\partial u_2} = 0$

$$\Rightarrow \frac{AE}{2l} \times 2 (u_2 - u_1) - \frac{q_0 l}{2} - F_2 = 0$$

$$\frac{AE}{l} (u_2 - u_1) = \frac{q_0 l}{2} + F_2 \qquad \text{... (1.34)}$$

Arrange the equation (1.33) and (1.34) in matrix form, we get,

$$\frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \frac{q_0 l}{2} \\ \frac{q_0 l}{2} \end{Bmatrix} + \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \text{... (1.35)}$$

$$[k]\{u\} = \{F\}$$

1.21.2. One-dimensional Heat Transfer Elements Based on the Stationary of a Functional

Consider a bar element with nodes 1 and 2 as shown in Fig.1.32. T_1 and T_2 are the temperatures at the respective nodes. So, T_1 and T_2 are considered as degrees of freedom of this bar element.

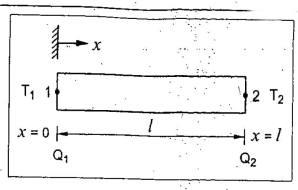


Fig. 1.32. Heat transfer element

We know that, $T(x) = N_1 T_1 + N_2 T_2$ ... (1.36)

$$T(x) = \left(1 - \frac{x}{l}\right) T_1 + \left(\frac{x}{l}\right) T_2 \qquad \text{... (1.37)}$$

$$\left[\because N_1 = 1 - \frac{x}{l};\ N_2 = \frac{x}{l}\right]$$

The strain energy stored within the element is given by,

$$U = \frac{1}{2} \int_0^l k \left(\frac{dT}{dx}\right)^2 dx \qquad \text{... (1.38)}$$

Potential energy of external force is given by

$$H = \int_0^l q_0 T\, dx + Q_1 T_1 + Q_2 T_2 \qquad \text{... (1.39)}$$

The total potential energy,

$$\pi = U - H$$

$$= \frac{1}{2} \int_0^l k \left(\frac{dT}{dx}\right)^2 dx - \int_0^l q\, T\, dx - Q_1 T_1 - Q_2 T_2 \qquad \text{... (1.40)}$$

From equation (1.37),

$$\frac{dT}{dx} = -\frac{1}{l} T_1 + \frac{1}{l} T_2$$

$$\frac{dT}{dx} = \frac{1}{l} (T_2 - T_1) \qquad \text{... (1.41)}$$

Substitute the equation (1.41) in equation (1.40),

$$\pi = \frac{1}{2} \int_0^l k \left[\frac{1}{l} (T_2 - T_1)\right]^2 dx - \int_0^l q_0 T\, dx - Q T_1 - Q T_2$$

(5) $\Rightarrow$ $$x_2 + \frac{12}{14}x_3 = \frac{60}{14}$$

$\Rightarrow$ $$x_2 + \frac{12}{14}(-2.77) = \frac{60}{14}$$

$\Rightarrow$ $$\boxed{x_2 = 6.66}$$

(4) $\Rightarrow$ $$x_1 + \frac{1}{4}x_2 - \frac{2}{4}x_3 = 0$$

$\Rightarrow$ $$x_1 + \frac{1}{4}(6.66) - \frac{2}{4}(-2.77) = 0$$

$\Rightarrow$ $$\boxed{x_1 = -3.05}$$

Result: $x_1 = -3.05$

$x_2 = 6.66$

$x_3 = -2.77$

Checking: Substituting x_1, x_2 and x_3 values in equation (1) or (2) or (3),

$\Rightarrow$ $$2x_1 + 4x_2 + 2x_3 = 15$$

$\Rightarrow$ $$2(-3.05) + 4(6.66) + 2(-2.77) = 15$$

So, our answer is correct.

1.24. ADVANTAGES OF FINITE ELEMENT METHOD

1. One of the major advantages of FEM over other approximate methods is the fact that FEM can handle irregular geometry in a convenient manner.
2. It handles general load conditions without difficulty.
3. Non-homogeneous materials can be handled easily.
4. All the various types of boundary conditions are handled.
5. Dynamic effects are included.
6. Vary the size of the elements to make it possible for using small elements where necessary.
7. Higher order elements may be implemented.
8. Altering the element model with different loads, boundary conditions and other changes in the model can be done easily and cheaply.

1.25. DISADVANTAGES OF FEM

✓ It requires a digital computer and fairly extensive software.

✓ It requires longer execution time compared with finite difference method.

✓ Output result will vary considerably, when the body is modeled with fine mesh when compared to body modeled with course mesh.

✓ In finite difference method, the governing differential equation of the phenomenon must be known whereas finite element method does not require to express fully.

1.26. APPLICATIONS OF FINITE ELEMENT ANALYSIS

The finite element can be used to analyse both structural and non-structural problems.

In structural problems, displacement at each nodal point is obtained. By using these displacement solutions, stress and strain in each element can be calculated.

Typical structural problems include:

1. Stress analysis including truss and frame analysis.
2. Stress concentration problems typically associated with holes, fillets or other changes in geometry in a body.
3. Buckling analysis: Example: Connecting rod subjected to axial compression.
4. Vibration analysis: Example: A beam subjected to different types of loading.

In non-structural problems, temperature or fluid pressure at each nodal point is obtained. By using these values, properties such as heat flow, fluid flow, *etc.*, for each element can be calculated.

Non-structural problems include:

1. Heat transfer analysis.

   Example: Steady state thermal analysis on composite cylinder.

2. Fluid flow analysis.

   Example: Fluid flow through pipes.

3. Distribution of electric or magnetic potential.

   Example: Modeling of electromagnetic field of motor.

Recently finite element analysis is used in some biomechanical engineering problems (which may include stress analysis) typically include analysis of human spine, skull, hip joints, heart, eye, *etc.*