Lecture Notes

Optimization in Engineering

[OE]

Module - I, II, III & IV

Semester : 6th

6th Semester		**Optimization in Engineering**	**L-T-P 3-0-0**	**3 Credits**

Module I: **(10 Hours)**
Idea of Engineering optimization problems, Classification of optimization algorithms, modeling of problems and principle of modeling. Linear Programming: Formulation of LPP, Graphical solution, Simplex method, Big-M method, Revised simplex method, Duality theory and its application, Dual simplex method , Sensitivity analysis in linear programming.

Module II: **(10 Hours)**
Transportation problems: Finding an initial basic feasible solution by Northwest Corner rule, Least Cost rule, Vogel's approximation method, Degeneracy, Optimality test, MODI method, Stepping stone method. **Assignment problems:** Hungarian method for solution of Assignment problems. Integer Programming: Branch and Bound algorithm for solution of integer programming problems.

Module III: **(12 Hours)**
Non-linear programming: Introduction to non-linear programming. Unconstraint optimization: Fibonacci and Golden Section Search method. Constrained optimization with equality constraint: Lagrange multiplier, Projected gradient method. Constrained optimization with inequality constraint: Kuhn-Tucker condition, Quadratic programming.

Module IV: **(6 Hours)**

Queuing models: General characteristics, Markovian queuing model, M/M/1 model, Limited queue capacity, multiple server, Finite sources, Queue discipline.

Optimization Engineering

Unit-1

Introduction :

It is a science that came into existence in a military content. During World War II, the military management of UK called on scientists from various disciplines and organized them into teams to assist it in solving strategic and tactical problems.

Definitions :

It is the basis how to maximise benefit and minimise cost.

Maximise (profit) (Benefit) — minimise (loss) (Cost)

* Operation Research is the scientific knowledge through interdisciplinary team efforts for the purpose of determining the best utilization of limited resources.

Scope of Operation Research :

Great scope for Economist, Statisticians, administrators and technicians.

Besides this, various other important field like -

(i) Agriculture
(ii) Industry
(iii) Research & Development
(iv) Finance
(v) Marketing
(vi) production Management
(vii) Military operations

Phases of Operation Research :

(i) Formulating the problems
(ii) Constructing a mathematical model
(iii) Deriving the solution from the model

(iv) Testing the model and updating the model
(v) Controlling the solution
(vi) Implementation

Models in Operation Research :

A model is a simplified representation of an operation.

A good model (Characteristics)

(i) The no. of variables used should be as few as possible.
(ii) The no. of assumptions should be as few as possible.
(iii) It should be easy and economical to construct.
(iv)

Classification of Models :

(1) Models by function -

(a) Descriptive Model - They describe and predict the facts and relationships among the various activities of the problem.

→ They do not have an objective function.

(b) Predictive Model - These models are used in predictive analysis involving a variety of statistical techniques used to analyze the current and historical facts.

(c) Normative or Optimization Models : They are prescriptive in nature and develop objective decision rule for optimum solutions.

(2) **Model by structure** :-
(1) physical Model
(2) Mathematical Model
(3) Analog Model

(1) **physical Model** -
It is the picturial representation of a real system.
Ex - Map blue print house

(2) **Mathematical Model** -
It represents a set of mathematical symbols to represent the Components of the real system. This Variables are related together by meanse of mathematical equation to describe the behaviour of the System.

(3) **Analog Model** -
This is a model in which one set of properties are used to represent another set of properties.

(3) **Model by Nature** :-
(1) Deterministic Model
(2) probabilistic Model

(1) **Deterministic Model** -
In this model all the parameters and functional relationship are assumed to be known with Certainity.
e.g - Linear programming problem

② probabilistic Model :-
In this Model at least one parameter of decission variable having random variable.

Linear Programing problem -

→ It is a mathematical technique to help Management decide how to make most effective use of organisation resources.

→ It deals with the optimization (Maximization or Minimization) of a function of a variable is known as objective function

Application of LPP -

1. Manufacturing problem
2. Assembly problem
3. Transportation problem.

Guidelines for Model Formulation :-

1) Understand the problem
2) Describe the objective into words
3) Describe the Constraints into Words
4) Define decission Variable
5) Write the objective interms of decission variable.
6) Write the Constraints interms of decission Variable.

Topic:

Linear Programming Problem

Linear Programing Problem (LPP)

Linear means variable having power 1.

→ Linear means 1st order, programming stand for planning.

→ LPP means the nature of problem or characteristics of the problem only of 1st order.

① Formulation -

→ LPP Consists of 3 different basic elements.

(a) Decission Variable
(b) Objective function
(c) Constraints

(a) Decission variable -

It is the unknown quantity ~~of~~ in a problem which is to be calculated by solving the problem.

(b) Objective function -

It is a mathematical expression which represents a linear function of the decission variable and the objective of the problem i.e to be maximized or minimized.

(c) Constraints -

Can be represented by finding
(i) Marketing Constituent or variable
(ii) Inter-relationship b/w variable.

It is the mathematical expression which represent the restriction or limits of the problem.

Ex:- Market Condn, inter relationship / betn the decision variable.

Non-Negative Constraint

Always ≥ 0

Mathematical Formulation of L.P.P :-

The mathematical formulation of LPP depends on the following steps -

→ (1) Write down the decission Variable.

i.e $x_1, x_2, x_3 \cdots\cdots x_n$

Consider 'n' no. of variables.

$c_1, c_2, c_3 \cdots\cdots c_n$ → Coefficient of decission Variable present in objective function.

→ (2) Formation of the objective function (maximize or minimize) using decission Variable.

Denoted by z.

$$\text{Max / min } z = c_1x_1 + c_2x_2 + c_3x_3 + \cdots c_nx_n$$

Max → profit
Min → Loss

→ (3) Subjected to Constrained (STC)
Represented by 'a'.

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n \; (\le, \ge, =) \; b_1$$

b_1 → R.H.S value of Constrained equn or Initial basic Solution

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \; (\le, \ge, =) \; b_m$$

$a_{11}, a_{12}, a_{13} \cdots a_{m1} \cdots a_{mn}$ → Coeff. of decission Variable w.r.t Constraint

→ (4) Consider the decission variable as non-negative value.

(1) A manufacture of wooden articles produce table and chair which requires 2 types of inputs i.e wood and labour. The manufacture known that for a table three unit of wood and one unit of labour are required. While for a chair there are 2 units for each input. The profit for each chair is 16 and each table is 20. The total available resource are 150 unit wood and 75 unit of labour. Manufacture wants to maximize his profit by distributing its resources for chair and table. Formulate the problem mathematically.

Sol^n- Let x_1 is no. of tables & x_2 is no. of chairs
profit from one table is Rs 20.
profit from one chair is Rs 16.

profit of x_1 no. of table is $20x_1$
" " x_2 no. of chair is $16x_2$.

step1 Decission Variable
x_1 = No. of units of tables
x_2 = No. of " chair

step2 Objective function
Maximize Z
$$\text{Max } Z = c_1x_1 + c_2x_2$$
$$\text{Max } Z = 20x_1 + 16x_2$$ (20 & 16 are related to profit)
Where ~~$x_1 + x_2$~~ >

step3 Subjected to Constraint (STC)
Wood require 3 unit for a table & 2 unit for a chair.
~~for x_1 no. of table~~ & x_2 no. of chair, wood ~~is required~~ ~~$3x_1$~~ -

	Table (x_1)	Chair (x_2)	
Wood	3	2	→ 150
Labour	1	2	→ 75

For x_1 no. of table & x_2 no. of chair wood is required - $3x_1 + 2x_2$

But 150 wood is available so,

$$3x_1 + 2x_2 \le 150$$

Similarly 1 labour required for a table and 2 labour required for a chair -

$$x_1 + 2x_2$$

But 75 labour is available so,

$$x_1 + 2x_2 \le 75$$

step 4 Non-nty constraint

$(x_1 \ge 0 \quad , \quad x_2 \ge 0) \quad , \quad x_1, x_2 \ge 0$

The required LPP for the above problem is

$$\boxed{\begin{aligned} \text{Max } Z &= 20x_1 + 16x_2 \\ \text{S.T} \quad 3x_1 + 2x_2 &\le 150 \\ x_1 + 2x_2 &\le 75 \\ (x_1, x_2 &\ge 0) \end{aligned}}$$

(Q) A farm manufacture produce 2 products A and B with a profit of Rs. 4 and Rs 5. The product are made by machine X and Y. The availability to the machine time in minutes for machine X and Y are 2000 and 1500 mins respectively. The time taken to make product A by machine X and Y are 5 and 4 mins and for product B are 3 and 4 mins. Formulate the LPP model to maximize the profit.

Ans **Decission Variable**

x_1 = Units of A

x_2 = units of B

Objective function

Maximize Z.

$$Max\ Z = C_1x_1 + C_2x_2$$

$$Max\ Z = 4x_1 + 5x_2$$

where $x_1, x_2 \geq 0$

Constraints

	$A(x_1)$	$B(x_2)$	
Machine x	5	3	2000
Machine y	4	4	1500

$$5x_1 + 3x_2 \leq 2000$$

$$4x_1 + 4x_2 \leq 1500$$

The required L.P.P for the above problem is

$$max\ z = 4x_1 + 5x_2$$

Subjected to -

$$\boxed{\begin{array}{l} 5x_1 + 3x_2 \leq 2000 \\ 4x_1 + 4x_2 \leq 1500 \\ \text{Where } x_1, x_2 \geq 0 \end{array}}$$

(Q)

(Q) Vitamin C and vitamin E are found in two different fruit F_1 and F_2. One unit of fruit F_1 Contains 3 units of ~~unit of~~ vitamin C and 2 units of vitamin E and fruit F_2 Contains 2 units of vitamin C and 3 unit of vitamin E. A patient need minimum 30 units of vitamin C and 20 units of Vitamin E. One unit of F_1 cost Rs 20 and one unit F_2 cost Rs 25. The hospital wants to purchase such units of Fruit F_1 & F_2 which should be supplied to the patient to meet the minimum requirement at minimum Cost. Formulate the problem into LPP.

Ans. Decission variable

x_1 is for fruit f_1
x_2 is for fruit f_2

Objective function

Minimise cost -

$$\min z = 20x_1 + 25x_2$$

One unit of f_1 cost 20
" " f_2 cost 25

Subjected to Constraint -

	$F_1(x_1)$	$F_2(x_2)$	
Vitamin C	3	2	30
" E	2	3	20

For Vitamin C, $3x_1 + 2x_2 \geq 30$
" E, $2x_1 + 3x_2 \geq 20$

The required L.P.P for the given problem is-

$$\boxed{\begin{aligned} &\min z = 20x_1 + 25x_2 \\ \text{s.t}\quad &3x_1 + 2x_2 \geq 30 \\ &2x_1 + 3x_2 \geq 20 \\ &(x_1, x_2 \geq 0) \end{aligned}}$$

(Q) A firm manufactures 3 products A, B and C. The profit are Rs 3, 2 and 4 respectively. The firm has 2 machines and given below is the required processing time in minutes for each machine on each product.

Machine	product wise processing time (min)		
	A	B	C
M_1	4	3	5
M_2	3	2	4

Machine M_1 and M_2 have 2000 and 2500 m/c minutes respectively. The firm must manufacture 100 units of A, 200 unit of B and 50 units of C but not more than 150 units of A. Set up an LPP to maximize the profit.

Ans Let Product $A = x_1$, $B = x_2$ and $C = x_3$.

Objective function -

$$\text{Max } z = 3x_1 + 2x_2 + 4x_3$$

Constraint -

for Machine M_1, $4x_1 + 3x_2 + 5x_3 \le 2000$

" M_2, $3x_1 + 2x_2 + 4x_3 \le 2500$

Since the firm manufactures 100 units of A, 200 units of B and 50 units of C but not more than 150 units of A, the further restrictions becomes,

$$100 \le x \le 150$$

$$200 \le x_2 \ge 0$$

$$50 \le x_3 \ge 0$$

Model summary -

$$\boxed{\begin{aligned} &\text{Max } z = 3x_1 + 2x_2 + 4x_3 \\ \text{S.T } \; &4x_1 + 3x_2 + 5x_3 \le 2000 \\ &3x_1 + 2x_2 + 4x_3 \le 2500 \\ &100 \le x \le 150, \; 200 \le x_2 \ge 0, \; 50 \le x_3 \ge 0 \end{aligned}}$$

Problem - 3: A company has three operational departments (i.e., ~~Wea~~ Weaving, Processing and Packing) with capacity to produce three different types of clothes namely suitings, shirtings, and woollens yielding a profit of ₹2, ₹4 and ₹3 per ~~minute~~ mitre respectively. One metre of suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing. Similarly one metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minute in packing. One metre of wollen requires 3 minutes in each department. In a week, total run time of each department is 60, 40, 80 hours for weaving processing and packing respectively. What is the linear programming problem model to find the product mix to maximize the profit ?

Solution: Let us designate

Weekly production of suitings $= x_1$ meteres

Weekly production of shirtings $= x_2$ metres

Weekly production of woolens $= x_3$ metres

Since it is not possible to produce negative quantities, feasible alternative are sets of values of x_1, x_2 and x_3 satisfying

$$x_1 \geq 0, \quad x_2 \geq 0, \text{ and } \quad x_3 \geq 0$$

The data of the given problem is summarized below:

Departments → / Type of Clothes ↓	Weavings (in Min)	Processing (in Min)	Packing (in Min)	Profit (₹/metre)
Suitings	3	2	1	2
Shirtings	4	1	3	4
Woolens	3	3	3	3
Availability (min)	60 X 60	40 X 60	80 X 60	

→ The key decision is to determine the weekly rate of production for the three types of clothes to maximize the profit.

$$\text{Max } Z = 2x_1 + 4x_2 + 3x_3$$

s.t. $\quad 3x_1 + 4x_2 + 3x_3 \leq 3600$

$$2x_1 + x_2 + 3x_3 \leq 2400$$

$$x_1 + 3x_2 + 3x_3 \leq 4800$$

$$x_1, x_2, x_3 \geq 0$$

Ans

Solution of L.P.P :-

There are two types of Method to find out the solution for LPP.

(i) Graphical Method
(ii) Simplex Method

Graphical Method :- (Generally used for two variables)

(i) Consider the Constraint for the L.P.P as an equality equation,

(ii) plot the equation in the graph in such a way that each equation represents a straight line of the graph.

(iii) Mark the region of
  (a) The Constraint is $\geq$ (greater than equal to) type mark the region above to straight line.
  (b) The Constraint is $\leq$ (less than equal to) type mark the region below the straight line.

(iv) After marking the region, a common region is produced i.e known as feasible region. The points which is responssible for Creating feasible region ~~is responsible for~~ are called as Corner points.

(v) Find the value of Z by putting the value of the Corner point.

(vi) Identify the optimal value of Z from the above Calculation and write the model Summary.

* That is in the Case of Maximization problem, the optimal point Corresponds to the Corner point at which the objective function has a maximum value, and in the Case of minimization, the optimal Solution is the corner point, which gives the minimum value for the objective function.

OE : Linear Programming Problems (LPP)

Linear Programming Problem (LPP) : LPP is a mathematical technique which involves the allocation of limited resources in a optimal manner, on the basis of a given criterion of optimality.

LPP deals with the optimisation (maximisation or minimisation) of a function of variables known as 'objective function'. It is subject to a set of linear equalities and/or inequalities known as 'constraints'.

A problem of the form

$$\text{Max or Min } Z = a_1x_1 + a_2x_2 + \cdots\cdots + a_nx_n \longleftarrow (i)$$

$$\text{Subject to} \quad \left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &\le b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n &\le b_n \end{aligned}\right\} \longleftarrow (ii)$$

$$\text{and} \quad x_j \ge 0, \quad j = 1, 2, \ldots\ldots, n \longleftarrow (iii)$$

is known as 'Linear Programming Problem (LPP)' where equ^n(i) is known as 'Objective function', equ^n(ii) is known as 'constraints' and equ^n(iii) is known as 'non-negative restrictions'.

Solution of LPP By Graphical Method

* Any set of x_j ($j = 1, 2, 3, \ldots$) that satisfies the constraint set of the LPP is known as solution to LPP.
* Any set of x_j ($j = 1, 2, 3, \ldots$) that satisfies the constraint set and non-negative restrictions is known as feasible solution to the LPP.
* Any set of x_j ($j = 1, 2, 3, \ldots$) that satisfies the constraint set, non-negative restrictions and optimizes (either maximize or minimize) to objective function is called an optimal feasible solution to the LPP.

Graphical Method

Simple Linear programming problems with two decision variable can be solved by "Graphical Method".

* If 'm' is the number of constraints in a linear programming with two variables x and y and non-negativity constraints $x \geq 0$, $y \geq 0$; the feasible region in the graphical solution will be surrounded by 'm+2' no.s of lines.

Types of Solutions of LPP

1. Unique optimal solution
2. Infeasible solution
3. Unbounded solution
4. Multiple optimal solution

Problem-1: Consider the following LPP:

Minimize $Z = 2x_1 - 10x_2$

Subject to
$$x_1 - x_2 \geq 0$$
$$x_1 - 5x_2 \leq -5$$
$$x_1, x_2 \geq 0$$

What is the optimum solution which can meet the constraints?

(A) -6 (B) -10

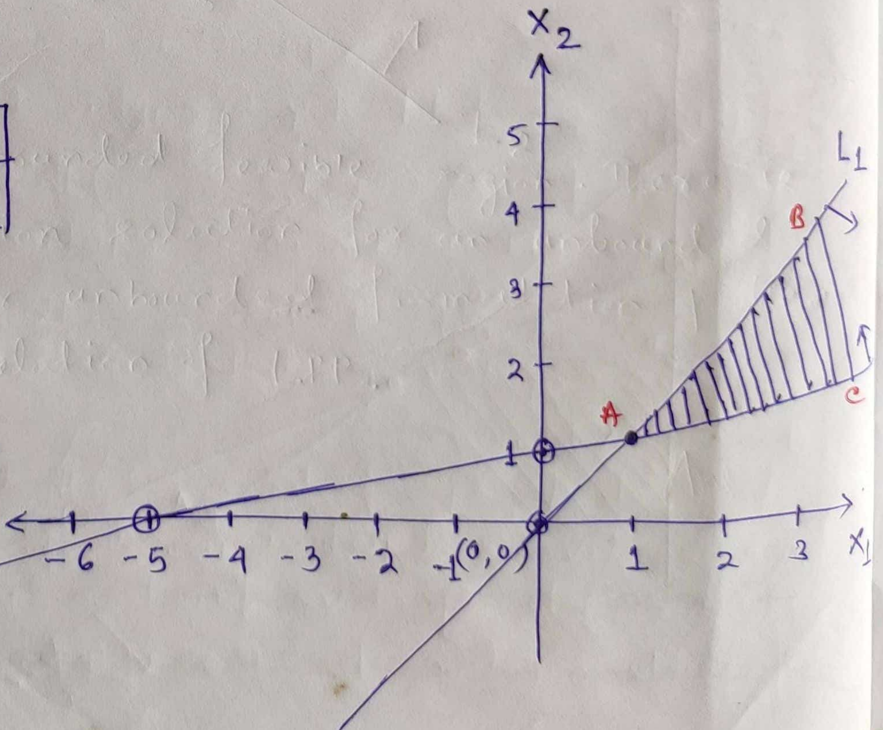
(C) -15 (D) -8

Solution:- $L_1: x_1 - x_2 = 0$

x_1	0	0
x_2	0	0

$L_2: x_1 - 5x_2 = -5$

x_1	0	-5
x_2	1	0



Here, it is cleared that, the feasible region is unbounded and lies in the $\angle BAC$. It is obivious that the minimum value of 'Z' occurs at point 'A' (1.25, 1.25).

Thus, $Z_{min} = 2 \times 1.25 - 10 \times 1.25 = -10$

Ans

Problem-2: Consider the LPP given in below:

$$\text{Maximize } Z = x_1 + x_2$$

Subject to constraints, $x_1 + x_2 \leq 1$

$$-3x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

(A) The LPP has a unique solution.

(B) The LPP has infeasible solution.

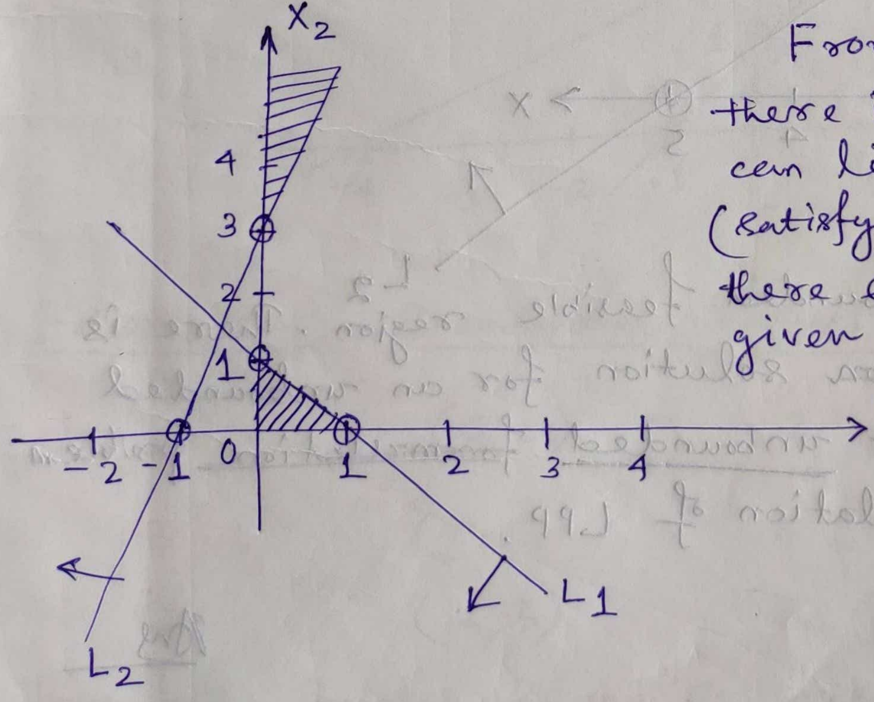
(C) The LPP has unbounded solution.

(D) The LPP has multiple optimal solution.

Solution:- $L_1: x_1 + x_2 = 1$ $\qquad$ $L_2: -3x_1 + x_2 = 3$

x_1	0	1
x_2	1	0

x_1	0	-1
x_2	3	0



From this graph, it is clear that there is no point (x_1, x_2) which can lie in both the regions (satisfy both the constraints), there exists no solution to the given problem. Hence there is infeasible solution.

Ans

Problem-3 : Consider the following Linear ~~Problem~~ Programming Problem (LPP) : Maximize, $Z = 3x + 2y$

Subject to

$$x \le 3$$

$$4x + 5y \ge 20$$

$$x, y \ge 0$$

(A) The LPP has a unique solution

(B) The LPP is infeasible

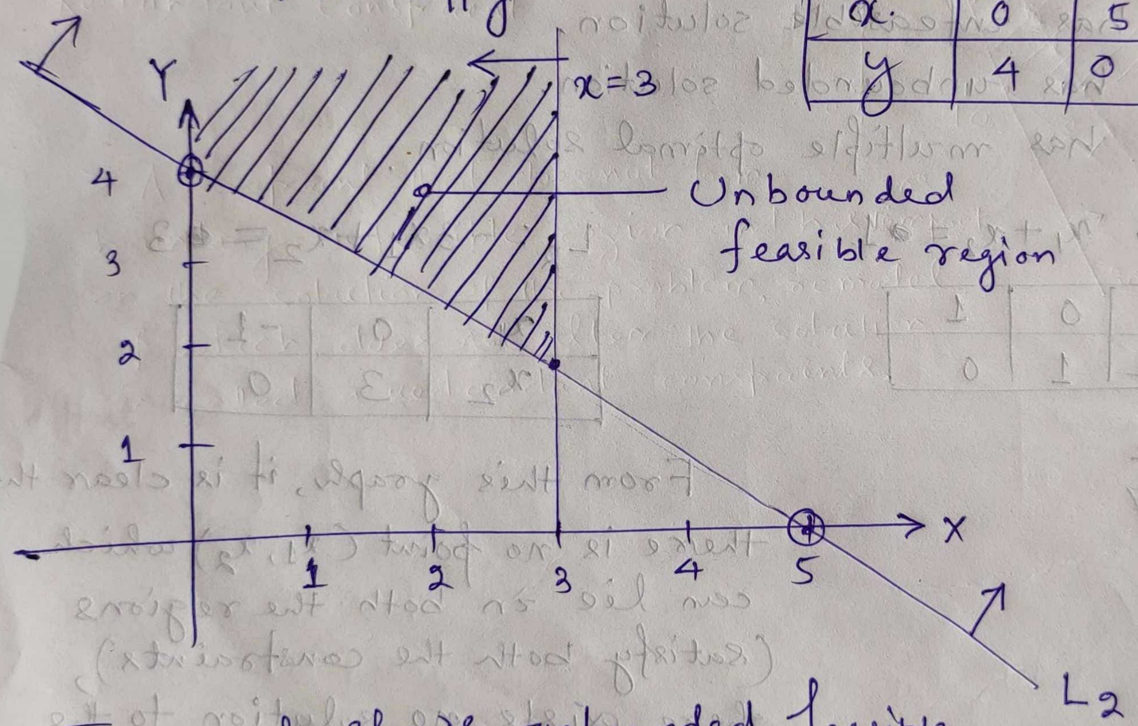
(C) The LPP is unbounded

(D) The LPP ~~is~~ has multiple optimal solution.

Solution:- $L_1 : x = 3$ i.e. $x \parallel y$

$L_2 : 4x + 5y = 20$

x	0	5
y	4	0



The graph shows unbounded feasible region. There is no single feasible ~~region~~ solution for an unbounded problem. The reason for unbounded ~~formulation~~ problem is incorrect formulation of LPP.

Ans

Problem-4 (Multiple Solution) Solve the following LPP Graphically,

$$\text{Max } Z = -x_1 + 2x_2$$

s.t. $x_1 - x_2 \le -1$

$-0.5x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

Solution:- R.H.S. of the 1^{st} constraint is negative. Multiplying both sides of the constraint by -1, it takes the form

$$-x_1 + x_2 \ge 1$$

$L_1: -x_1 + x_2 = 1$

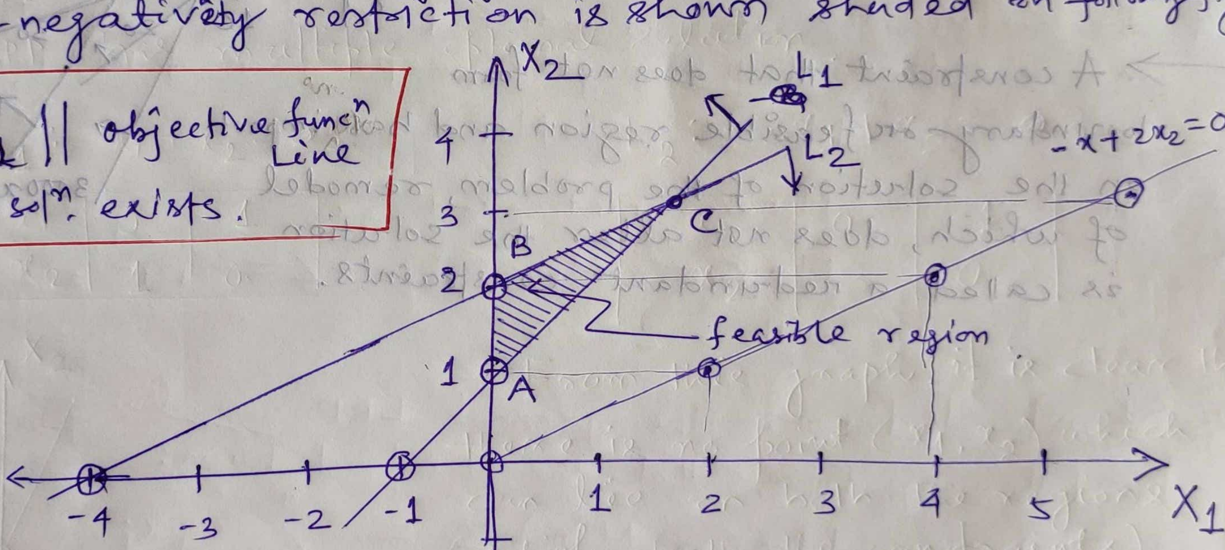
x_1	0	-1
x_2	1	0

$L_2: -0.5x_1 + x_2 = 2$

x_1	0	-4
x_2	2	0

The solution space satisfying the constraints and meeting the non-negativity restriction is shown shaded in following fig.

Here, $L_1 \parallel$ objective func^n Line
$\Rightarrow$ Multiple sol^n exists.



Extreme Point (vertex)	Co-ordinates	Values of Z
A	(0, 1)	2
B	(0, 2)	4 (Max^m)
C	(2, 3)	4 (Max^m)

objective function

Max $Z = -x_1 + 2x_2$

Taking $Z = 0$

$-x_1 + 2x_2 = 0$

x_1	0	2	4	6
x_2	0	1	2	3

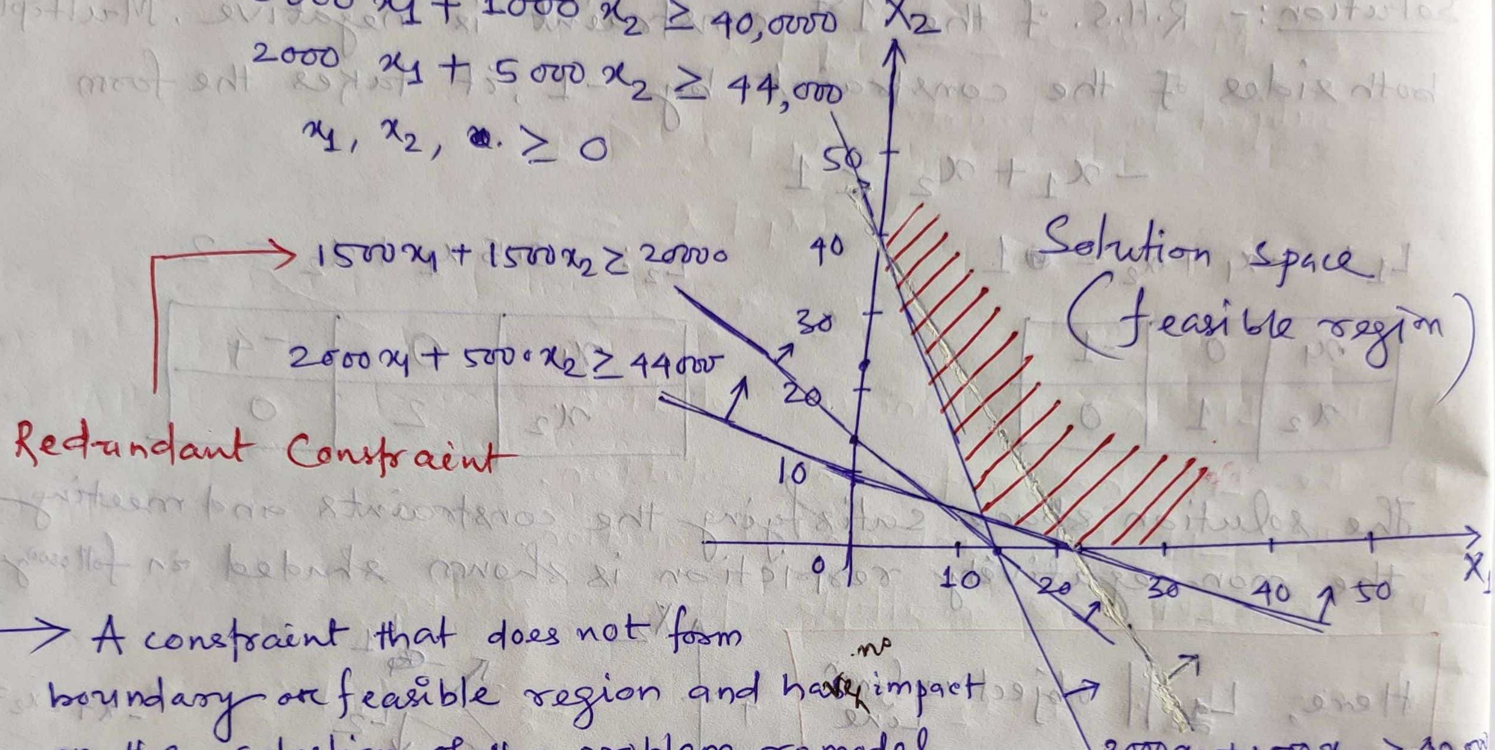
Thus both points B and C give the same maximum value of $Z = 4$. It follows that every point between B and C on the line BC also gives the same value of 'Z'. The problem, therefore has multiple optimal solution and $Z_{max} = 4$

* **Redundant Constraints** :- The constraints which does not affect the solution space (i.e. feasible region) is known as redundant constraints.

Ex:- $\text{Min } Z = 600x_1 + 400x_2$

s.t.

$$1500x_1 + 1500x_2 \geq 20000$$
$$3000x_1 + 1000x_2 \geq 40{,}000$$
$$2000x_1 + 5000x_2 \geq 44{,}000$$
$$x_1, x_2 \geq 0$$



→ A constraint that does not form boundary of feasible region and has no impact on the solution of the problem, remodel of which, does not alter the solution is called a redundant constraints.

Eg - Solve the following LPP by graphical Method -

$$\text{Max } Z = 20x_1 + 25x_2$$

S.T $2x_1 + 3x_2 \le 30$

$2x_1 + x_2 \le 20$

$x_1, x_2 \ge 0$

Soln

$2x_1 + 3x_2 = 30$ —— (I)

$2x_1 + x_2 = 20$ —— (II)

For equn (I)

put $x_1 = 0 \quad \therefore x_2 = 10$

for $x_2 = 0 \quad x_1 = 15$

x_1	0	15
x_2	10	0

$2x_1 + 3x_2 = 30$ passes through (15,10)

For equn (II) -

x_1	0	10
x_2	20	0

(10,20)

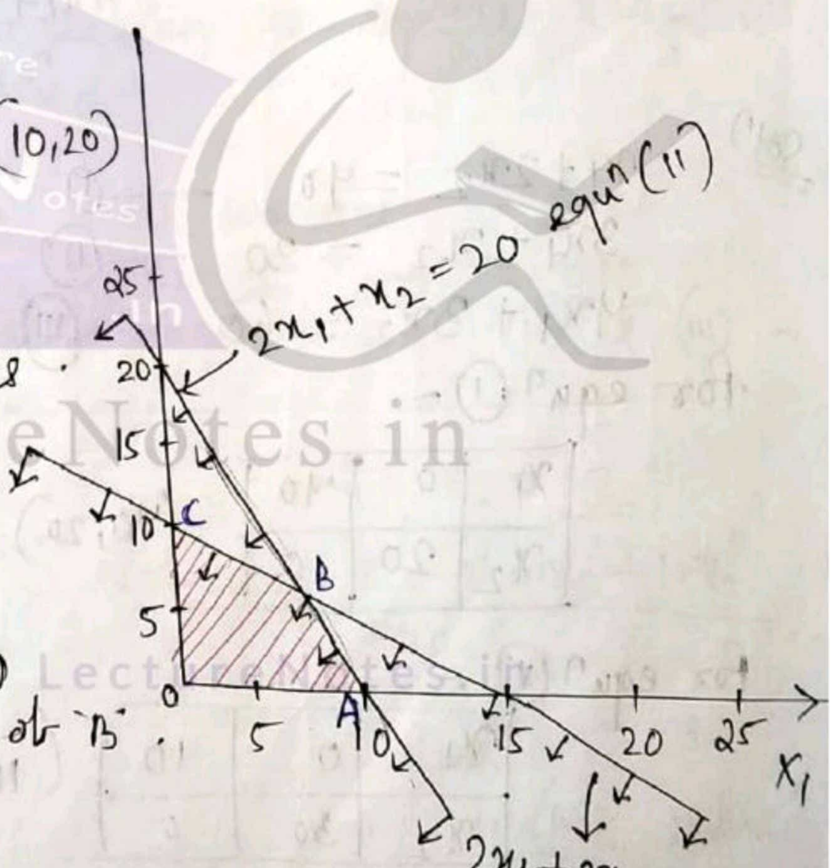
passes through (10,20)

→ A, B, C & O are corner points.

'B' is the point intersection of equn (I) & (II) -

By solving equn (I) & (II) We can get the value of 'B'

$$\begin{aligned} 2x_1 + 3x_2 &= 30 \\ \cancel{2x_1} + x_2 &= 20 \\ (-) \quad (-) \quad & (-) \\ \hline 2x_2 &= 10 \\ x_2 &= 5 \\ x_1 &= 7.5 \end{aligned}$$



* The feasible region is OABC.

Corner point	Value of $Z = 20x_1 + 25x_2$
O (0,0)	0
A (10,0)	200
B (7.5,5)	275 (Max. value)
C (0,10)	250

$\therefore$ The maximum value of Z occurs at $B(7.5, 5)$
Hence optimal solution is $x_1 = 7.5$ and $x_2 = 5$

(Q) Solve the following LPP by graphical method.

minimize $Z = 20x_1 + 10x_2$

Subject to, $x_1 + 2x_2 \le 40$

$3x_1 + x_2 \ge 30$

$4x_1 + 3x_2 \ge 60$

$x_1, x_2 \ge 0$

Sol^n^

$x_1 + 2x_2 = 40$ —— (i)

$3x_1 + x_2 = 30$ —— (ii)

$4x_1 + 3x_2 = 60$ —— (iii)

for equ^n^ (i) -

x_1	0	40
x_2	20	0

(40, 20) Equ^n^ (i) passes through (40, 20)

for equ^n^ (ii) -

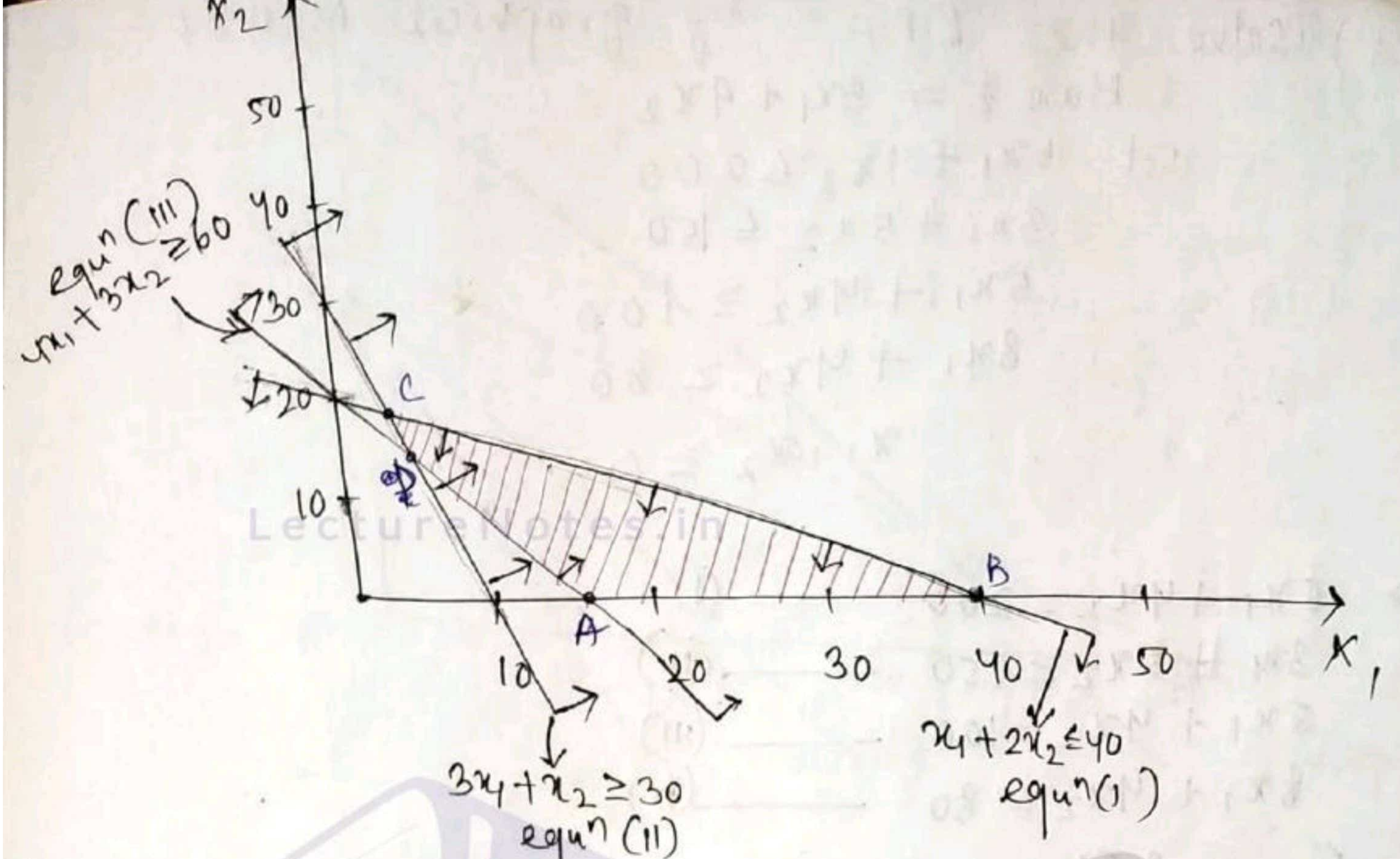
x_1	0	10
x_2	30	0

(10, 30)

for equ^n^ (iii) -

x_1	0	15
x_2	20	0

(15, 20)



point 'C' is the intersection of equ^n (i) & equ^n (ii)

$$x_1 + 2x_2 = 40$$
$$3x_1 + x_2 = 30$$

By solving (i) & (ii) we get

$x_1 = \underline{4}$ and $x_2 = \underline{18}$

point 'D' is the intersection of equ^n (ii) & (iii) -

By solving we get -

$x_1 = \underline{6}$ and $x_2 = \underline{12}$

Corner point	Value of $Z = 20x_1 + 10x_2$
A (15,0)	300
B (40,0)	800
C (4,18)	260
D (6,12)	240 (Min. value)

$\therefore$ The minimum value of Z occurs at D(6,12). Hence optimal solution is $x_1 = 6$ and $x_2 = 12$.

(Q) Solve the L.P.P by graphical Method.

$$\text{Max } z = 3x_1 + 4x_2$$

$$\text{s.t } 5x_1 + 4x_2 \le 200$$

$$3x_1 + 5x_2 \le 150$$

$$5x_1 + 4x_2 \ge 100$$

$$8x_1 + 4x_2 \ge 80$$

$$x_1, x_2 \ge 0$$

Ans

$5x_1 + 4x_2 = 200$ ———— (I)

$3x_1 + 5x_2 = 150$ ———— (II)

$5x_1 + 4x_2 = 100$ ———— (III)

$8x_1 + 4x_2 = 80$ ———— (IV)

for equ^n (I) -

x_1	0	40
x_2	50	0

(40, 50) Equ^n (I) passes through (0, 40) & (50, 0) (~~40, 50~~)

For equ^n (II) -

x_1	0	50
x_2	30	0

(50, 30)

for equ^n (III) -

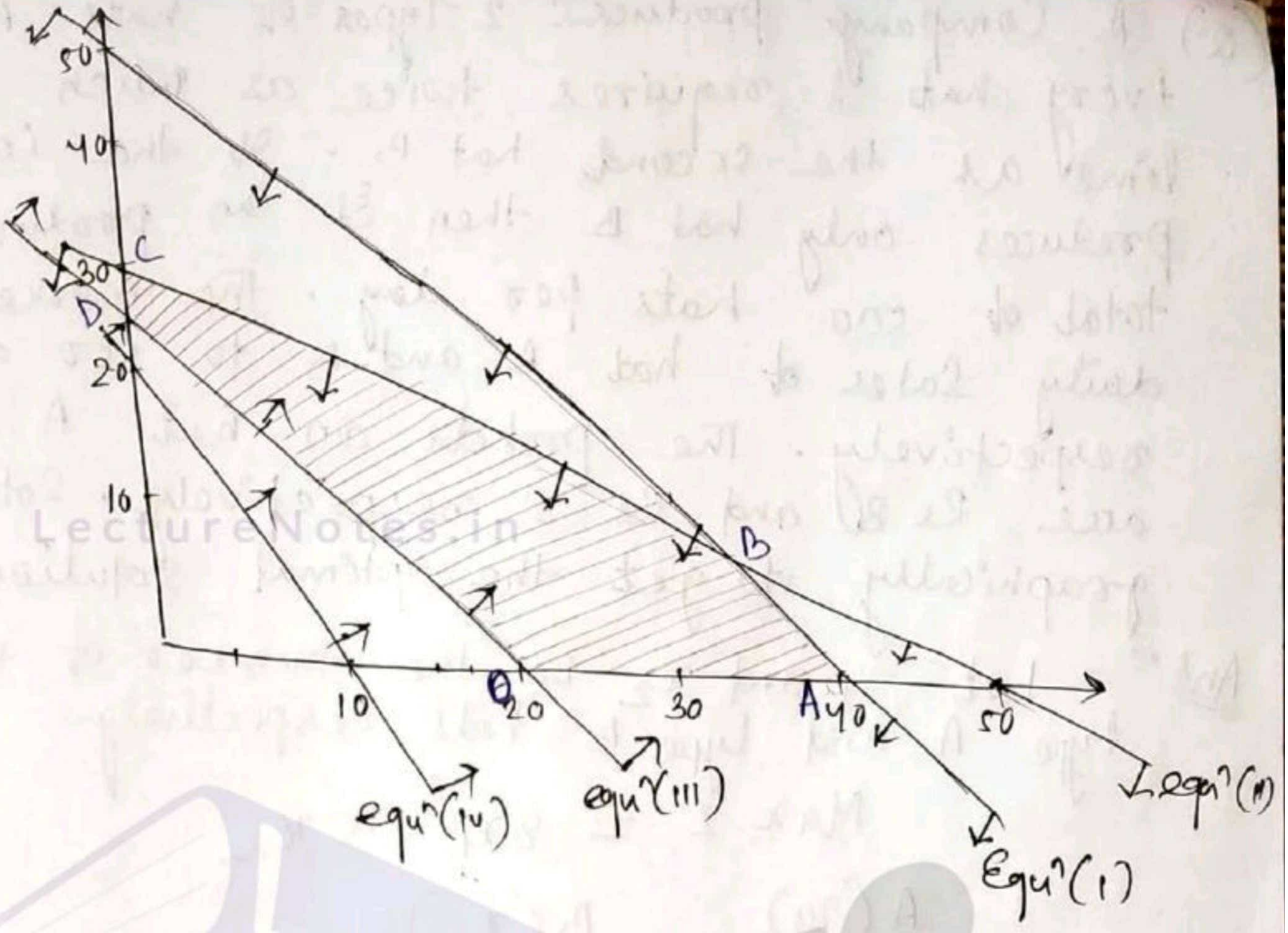
x_1	0	20
x_2	25	0

(20, 25)

for equ^n (IV) -

x_1	0	10
x_2	20	0

(10, 20)



Feasible region is given by OABCD.
point B is intersection of equ^n (i) & (ii).
By solving (i) & (ii) we get,

$$x_1 = 30.8$$
$$x_2 = 11.5$$

Corner point	Value of $z = 3x_1 + 4x_2$
O (20,0)	60
A (40,0)	120
B (30.8, 11.5)	138.4 (Max. value)
C ~~(30,0)~~ (0,30)	120
D (0, 25)	100

The maximum value of z occurs at B(30.8, 11.5)
The optimal solution is $x_1 = 30.8$ and $x_2 = 11.5$.

(Q) Use Graphical Method to solve the LPP

Unbounded sol^n

Maximize $z = 3x_1 + 2x_2$

s.t, $5x_1 + x_2 \geq 10$

$x_1 + x_2 \geq 6$

$x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

Sol^n

$5x_1 + x_2 = 10$ —— (i)

$x_1 + x_2 = 6$ —— (ii)

$x_1 + 4x_2 = 12$ —— (iii)

$x_1, x_2 \geq 0$ —— (iv)

Equ^n-(i)

$x_1 = 0$, $x_2 = 10$ $(0,10)$ $(2,10)$

$x_2 = 0$ $x_1 = 2$ $(2,0)$

Equ^n-ii

$x_1 = 0$, $x_2 = 6$ $(0,6)$ $(6,6)$

$x_2 = 0$, $x_1 = 6$ $(6,0)$

Equ^n-iii

$x_1 = 0$, $x_2 = 3$ $(0,3)$

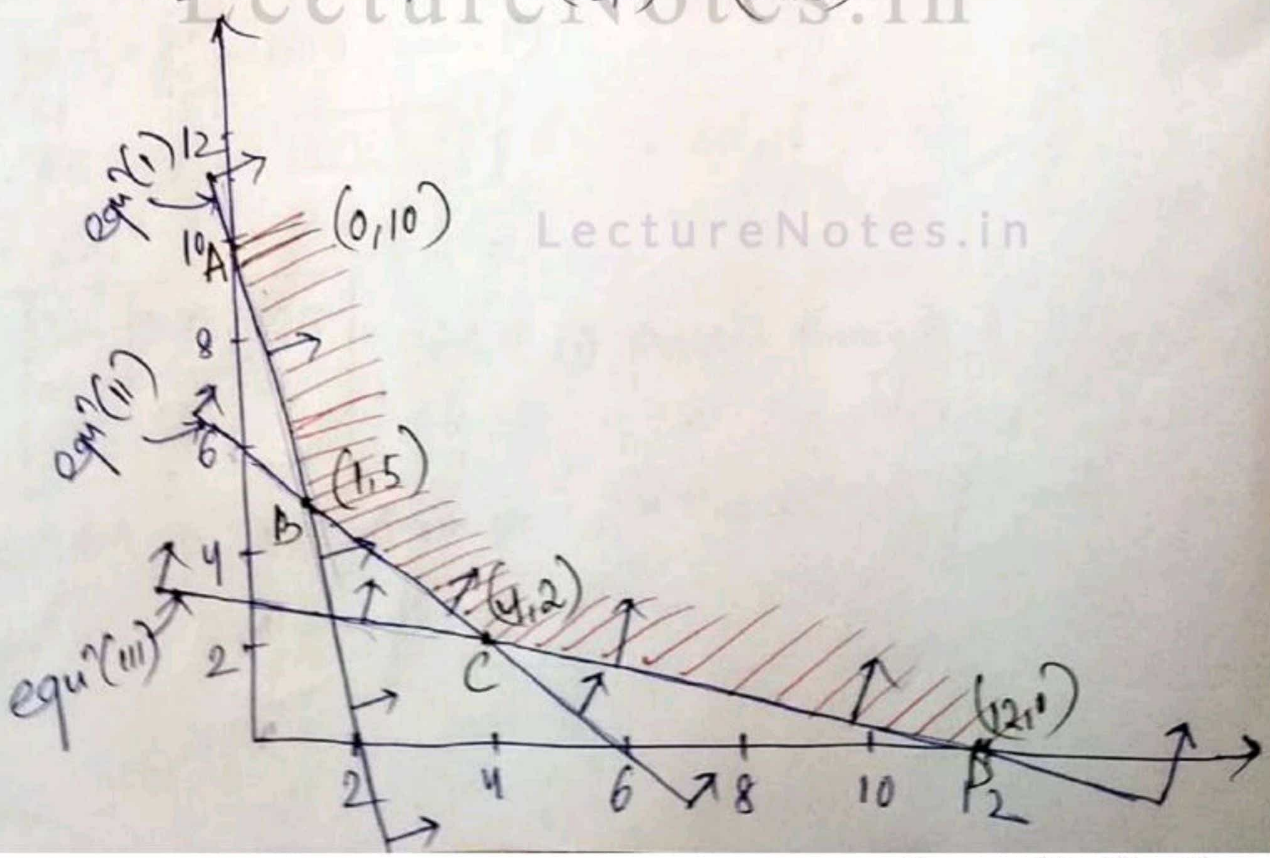
$x_2 = 0$, $x_1 = 12$ $(12,0)$ $(12,3)$

~~Equ^n-iii~~

~~$x_1 = 0$, $x_2 = 3$~~

~~$x_2 = 0$, $x_1 = 12$~~

~~$(12,0)$~~ $(0,3)$



Corner points	Value of $Z = 3x_1 + 2x_2$
A (0,10)	20
B (1,5)	13 (Min. Value)
C (4,2)	16
D (12,0)	36

Since the minimum value is attained at B(1,5) the optimum solution is $x_1 = 1$, $x_2 = 5$

Note: When in a fesible region there is no boundary is present then it is called unbounded sol^n. We can get ∞ no. of solution.

(1) In the above problem if the objective function is maximization, then the solution is unbounded as the max. value of Z occurs at infinity.

Some More Cases -

There are some linear programming problems which may have,

(i) a unique optimal solution
(ii) An infinite number of optimal solution / alternate optimal sol^n.
(iii) An unbounded solution.
(iv) No solution - Infesible sol^n.

(2) If previous question is in minimization we can find the optimal solution. Because our min value is at B (1,5).

Corner point	Value $Z = 3x_1 + 2x_2$
A (0, 10)	20
B (1,5)	13 (Min value)
C (4,2)	16
D (12,0)	36

Q Solve the LPP by graphical method –

Infinite no. of solution / Alternate solution

Maximize $Z = 100x_1 + 40x_2$

S.t $5x_1 + 2x_2 \le 1{,}000$

$3x_1 + 2x_2 \le 900$

$x_1 + 2x_2 \le 500$

$x_1, x_2 \ge 0$

Soln equn-I

x_1	0	200
x_2	500	0

(200, 500)

equn-II

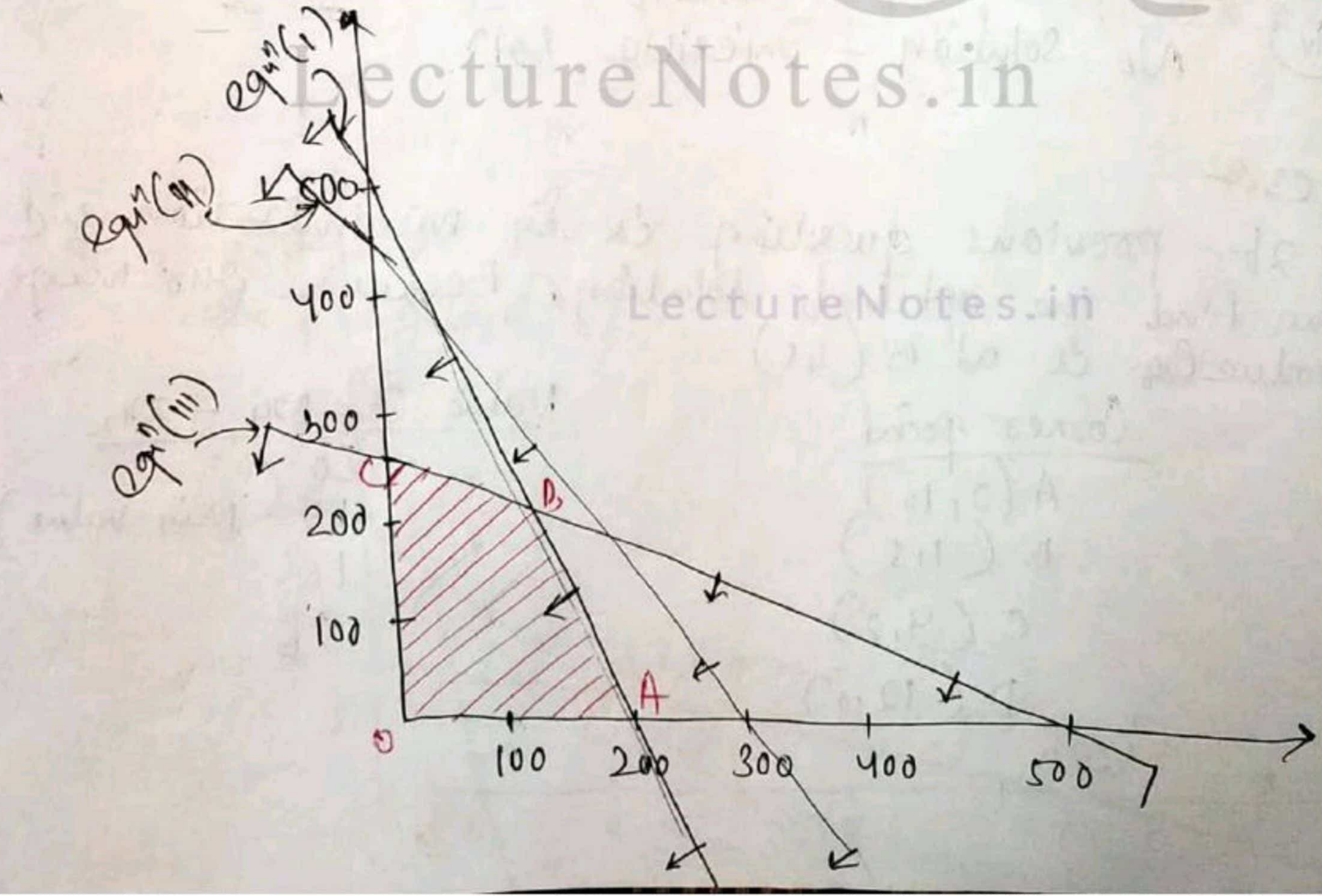
x_1	0	300
x_2	450	0

(300, 450)

equn-III

x_1	0	500
x_2	250	0

(500, 250)



OABC is the feasible region.

Corner points	Value of $Z = 100x_1 + 40x_2$
A (200, 0)	20000 (Max)
B (125, 187.5)	20000 (Max.)
C (0, 250)	10000

The Max. value occurs at point A and B. Further it is called alternate optimal solution. (Infinite no. of solution)

Infeasible Solution

Q) Solve the following L.P.P.

$$\text{Max } Z = x_1 + x_2$$

$$\text{s.t} \quad x_1 + x_2 \le 1$$

$$-3x_1 + x_2 \ge 3$$

$$x_1, x_2 \ge 0$$

sol^n equ^n-I

x_1	0	1
x_2	1	0

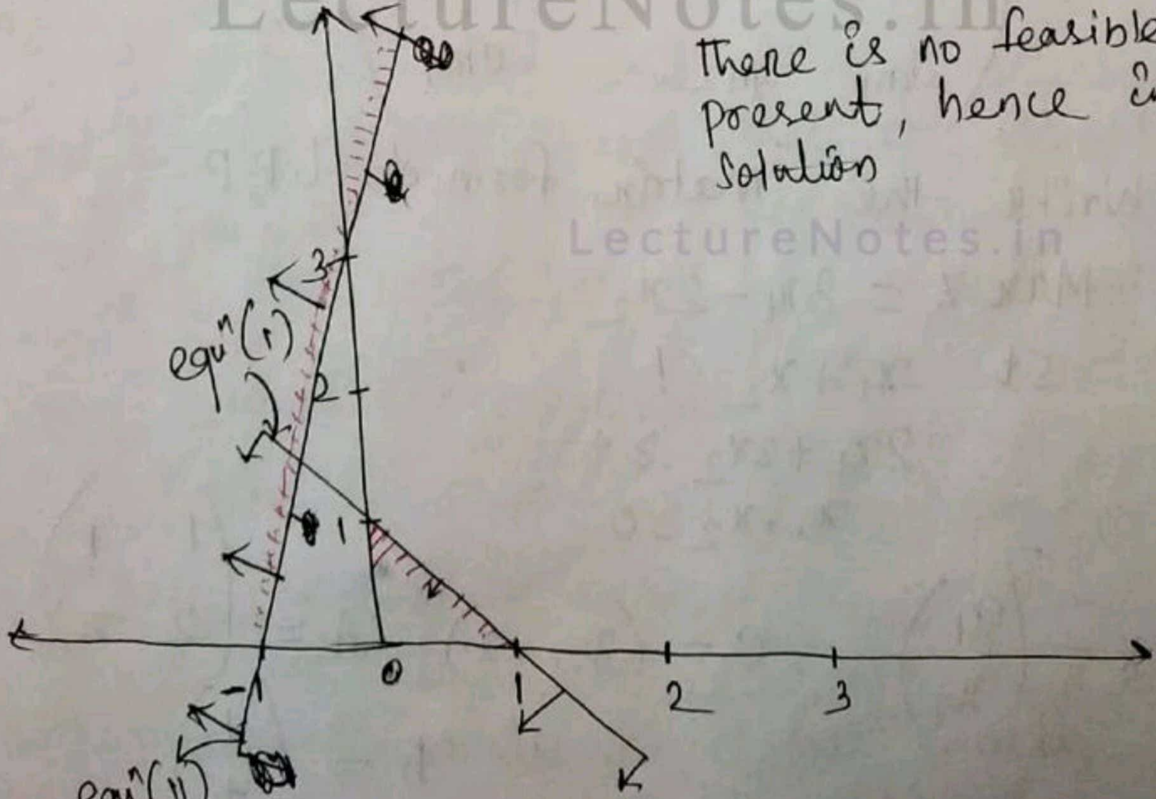
(1,1)

equ^n-II

x_1	0	−1
x_2	3	0

(−1,3)

There is no feasible region present, hence infeasible solution



* Formulation of LPP :

Problem-1 : A company makes two kinds of leather belts. Belt-A is a high quality belt and belt-B is of the lower quality. The respective profit are Rs. 4 and Rs. 3 per belt. Each belt of type 'A' requires twice as much time as a belt of type-B, and if all belts were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 belts per day. But 'A' requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day available for belt 'B'. What will the optimal product mix

Solution :- From question,

$$\text{Maximize } Z = 4x_1 + 3x_2$$

subject to constraints:

$$2x_1 + x_2 \le 1000 \quad \text{(Time constraints)}$$

$$x_1 + x_2 \le 800 \quad \text{(Availability of leather)}$$

$$\left.\begin{aligned} x_1 &\le 400 \\ x_2 &\le 700 \end{aligned}\right\} \text{(Availability of buckles)}$$

$$x_1, x_2 \ge 0$$

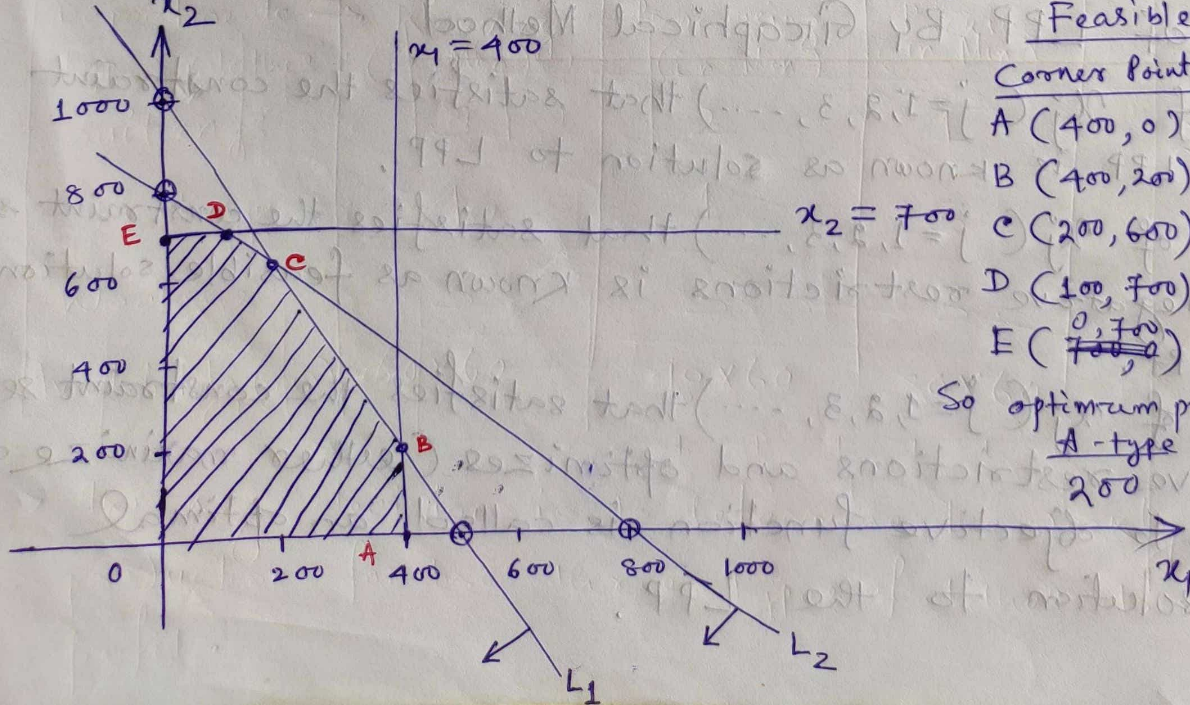
$L_1 : 2x_1 + x_2 = 1000$

x_1	0	500
x_2	1000	0

$L_2 : x_1 + x_2 = 800$

x_1	0	800
x_2	800	0

$x_1 = 400$ i.e. $x_1 \parallel x_2$ $\qquad$ $x_2 = 700$ $\quad x_2 \parallel x_1$



Feasible Region

Corner Points	Value of $Z = 4x_1 + 3x_2$
A (400, 0)	1600
B (400, 200)	2200
C (200, 600)	2600 (Max^m)
D (100, 700)	2500
E (~~700, 0~~ 0, 700)	2100

So optimum product Mix will be

A-type	B-type
200	600 Ans

Problem-2: A company produces two types of toys: P & Q. Production time of Q is twice that of P and the company has a maximum of 2000 time units per day. The supply of raw material is just sufficient to produce 1500 toys (of any type) per day. Toy type Q requires an electric switch which is available @600 pieces per day only. The company makes a profit of Rs 3 and Rs 5 on type P and Q respectively. For maximisation of profits, the daily production quantities of P and Q toys should respectively be

(a) 1000, 500 (b) 500, 1000

(c) 800, 600 (d) 1000, 1000

Solution: From the question, $\text{Max } Z = 3P + 5Q$

Subject to constraints:

$$P + 2Q \leq 2000$$
$$P + Q \leq 1500$$
$$Q \leq 600$$
$$P, Q \geq 0$$

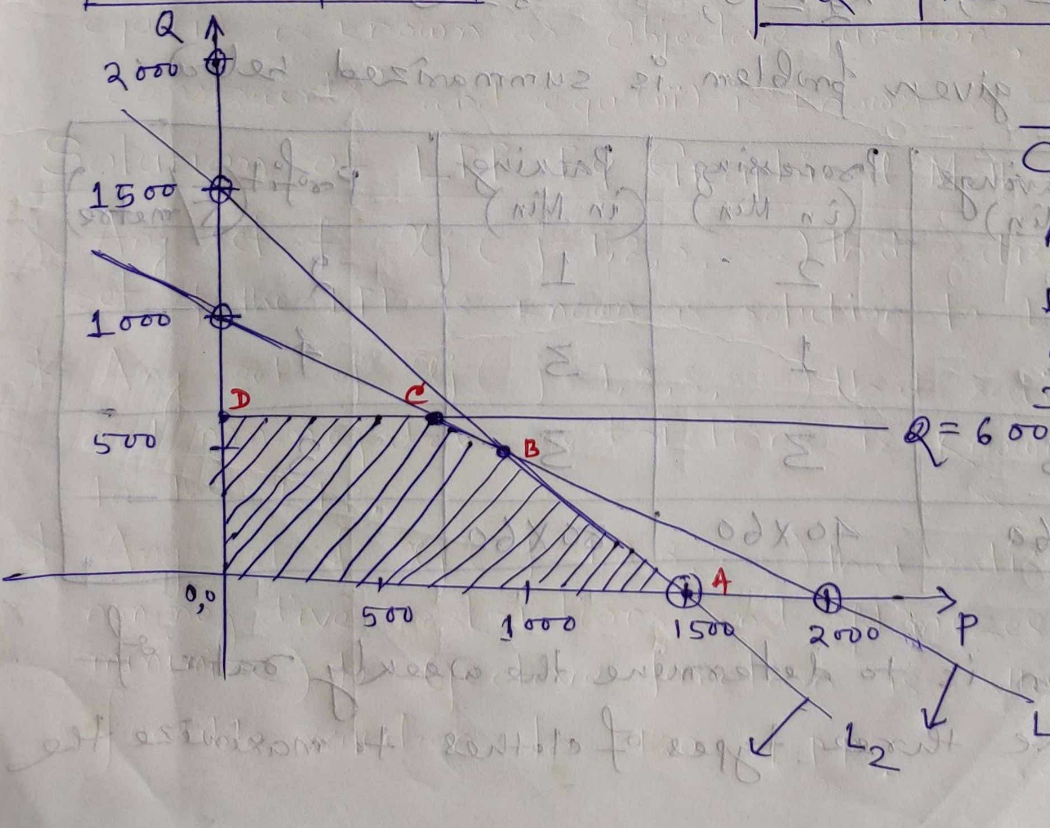
$L_1: P + 2Q = 2000$

P	0	2000
Q	1000	0

$L_2: P + Q = 1500$

P	0	1500
Q	1500	0

$Q = 600$ i.e. $Q \parallel P$



Feasible Region

Corner Points	Value of 'Z'
A (1500, 0)	4500
B (1000, 500)	10,500 ✓
C (800, 600)	5400
D (0, 600)	3000

Hence production quantity of P & Q are
P = 1000
Q = 500 Ans

Matrix form of LPP -

Maximize or minimize $\boxed{Z = CX}$ (objective fun^n)

C → Coef. of O.F

X → Decision Variable

Subjected to $\boxed{AX \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b}$

A → Coef. of Constraint

b → Right hand side Value

and $\boxed{X \ge 0}$

Where,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \qquad C = (C_1, C_2, C_3 \cdots C_n)$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \& \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Ex Write the Matrix form of L.P.P -

Max $Z = 3x_1 - 2x_2$

s.t $x_1 + x_2 \le 1$

$2x_1 + 2x_2 \ge 4$

$x_1, x_2 \ge 0$

Ans $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $C = (3, -2)$, $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$b = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Types of Solution -

① Solution - A set of Variables $x_1, x_2, x_3 \cdots x_n$ that satisfied the Constraint equation of LPP is called as solution.

② Feasible solution - Any solution of L.P.P which satisfied the constrained equation as well as non negativity restriction.

③ Optimum Solution - Any feasible solution, which optimizes (maximize or minimize) the objective function of L.P.P is known as optimum solution.

④ Basic Solution - If a system is having 'M' linear equation with 'n' variable $(M < n)$ any solution is obtained by solving for 'M' variables keeping the remaining $(n-m)$ variable zero(0) is called basic solution.

Such 'M' variable is called basic variable and the remaining is called as non-basic variable.

⑤ Basic feasible Solution -

It is the basic solution that satisfy the non-negative restriction and that is called basic feasible solution.

It is of 2 types -

(1) Non - degenerate

(2) Degenerate

(1) Non - degenerate -

A basic feasible solution is said to be non - degenerate if it has exactly 'M' positive $x_i(x_1, x_2 \cdots x_m)$ Variables.

(5) <u>Degenerate</u> - A basic feasible solution is said to be ~~non~~ degenerate if one or more basic variables are zero '0'.

(6) <u>Unbounded Solution</u> -

If the value of the objective function is maximizes or minimizes indefinitely is called unbounded solution.

→ There is no boundaries ~~and~~

Laws/Rules in LP (or) Assumption of LPP (Linear Programming Problem)

(1) Linearity or Proportionality -

The basic assumption of LP is linearity or proportionality exist in the objective function and constraint equation.

Ex - If we will sell a product which is having a profit of rupees 10 by selling 10 such product we can get the profit of rupees 100. That may not be always true because of quantity discount that is the selling price may be constant but the manufacturing cost will very with no. of unit produced. So the profit will very per unit.

1 product → Rs 10/- profit

10 product → Rs 10×10 profit

(2) Additivity -

Ex If we use T_1 hours in the machine A in order to produce product-1 and T_2 hours to produce product-2. Then the total time taken by the machine A is $T_1 + T_2$. If we change over the time from Product 2 to product 1 then the total time taken by the Machine 'A' is Same (T_2+T_1)

3. Certainity -

In all LPP programming model it is assume that all the model parameter such as availability of resources, profit contribution of a unit of decission variable and Consumption of Resources, constraint must be known and constant. Des not change with time

4. Divisibility -

The solution values of Decission Variable are assumed as either number or mixed number.

Forms of LPP -

(i) Canonical form (ii) Standard form

Canonical form	Standard form
→ The objective function may be maximised or minimised.	→ The objective function is always maximization type.
→ All Constraints are either "≤" or "≥" type	→ The constraints are '=' type
→ Decission Variable always ≥0.	→ Decission variable ≥ 0.
→ the Right hand side of all constraints are +ve.	→ the Right hand side of all constraints are +ve.

Algorithm

$$\text{Max (or) Min } Z = \sum_{j=1}^{n} C_j x_j$$

$$\text{s.t } \sum_{j=1}^{n} a_{ij} x_j \ (\le \text{(or)} \ge) \ b_i \quad (i=1,2,\dots m)$$

$$\text{and } x_j \ge 0 \ (j=1,2,3\dots n)$$

$$\text{Max } Z = \sum_{j=1}^{n} C_j x_j$$

$$\text{s.t } \sum_{j=1}^{n} a_{ij} x_j = b_i \ (i=1,2\dots n)$$

$$\text{and } x_j \ge 0 \ (j=1,2,\dots n)$$

Note * Before Converting LPP to Standard forms we should check whether it is Canonical or not.

* If the given objective function is in minimization form convert to maximization by multiplying (-1)

eg - $\min z = 5x_1 + 6x_2 \Rightarrow$

$\Rightarrow -(\min z) = -5x_1 - 6x_2$

$\Rightarrow \max z = -5x_1 - 6x_2$

Slack Variable -

A non-negative variables which when added ~~too~~ ~~a~~ in the left hand side of the Constraint to convert less than equal ($\leq$) type to ~~making~~ into equality is called ~~Surplus~~ slack variable.

eg - $2x_1 + 3x_2 \leq 20$

$$2x_1 + 3x_2 + s_1 = 20$$

s_1 → Slack variable

Surplus Variable -

A non-negative variable which when substracted from the Left hand side of Constraint equation to convert the greater than equal ($\geq$) into equality is called Surplus variable.

eg $2x_1 + 4x_2 \geq 40$

$$2x_1 + 4x_2 - s_2 = 40$$

s_2 → Surplus variable

Algorithm for Conversion of Canonical form to standard form :-

1. Check the objective function is of min type or max type. If the OF is min type convert into max. type by multipling (-ve) on both side.

eg $\text{Min } z = 5x_1 + 6x_2$

$\Rightarrow -(\text{min } z) = -5x_1 - 6x_2$

$\Rightarrow \text{Max } z = -5x_1 - 6x_2$

2. Check the Constraints are equal to type or not.
   (a) If the constraint equations are '≤' type add some slack variable.
   (b) If the constraint equations are '≥' type substract some surplus variable ~~and add some artificial variable~~

3. Check whether the right hand side value of the Constraints are (+ve) or not. If not (+ve), then convert them to (+ve) value by multiplying (-ve) sign.

4. Consider the decission variable non-negative.

(Q) Convert the canonical form to standard form.

$$\text{Min } z = x_1 + 10x_2 + x_3$$

$$\text{s.t} \rightarrow 3x_1 + x_2 + 3x_3 \le -5 \quad \text{--- (1)}$$

$$x_1 + x_2 + 6x_3 \le 15 \quad \text{--- (2)}$$

$$6x_1 + 5x_2 + 3x_3 \ge 22 \quad \text{--- (3)}$$

Ans

Step-I : Change objective function to maximize type.

$$\text{Max } z = -(\text{Min } z)$$

$$\text{Max } z = -(x_1 + 10x_2 + x_3)$$

Step-II : Given $3x_1 + x_2 + 3x_3 \le -5$

$$-(3x_1 + x_2 + 3x_3) \ge 5$$ Multiply (-ve) Sign

Step - III   As R.H.S values are '≥' type, to make the constraint into '=' type we substract some value from L.H.S.

$$-(3x_1 + x_2 + 3x_3) - s_1 = 5$$

$$x_1 + x_2 + 6x_3 + s_2 = 15$$

$$6x_1 + 5x_2 + 3x_3 - s_3 = 22$$

Where, $x_1, x_2, x_3 \geq 0$, $s_1, s_2, s_3 \geq 0$

Now Required L.P.P is –

Represent the new variables in the o.f and the coeft. of slack variable is zero

$$\text{Max } z = -x_1 - 10x_2 - x_3 + 0s_1 + 0 \cdot s_2 + 0 \cdot s_3$$

s.t.

$$-(3x_1 + x_2 + 3x_3) - s_1 = 5$$

$$x_1 + x_2 + 6x_3 + s_2 = 15$$

$$6x_1 + 5x_2 + 3x_3 - s_3 = 22$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Topic:

Simplex Method

SIMPLEX METHOD

This method is used for two or more than two decission variables present in the LPP.

→ Simplex alogorithm is an iterative procedure for solving LP problems. ~~It consists of~~ -

(i)

* All constraint should be $\le$ if $\ge$ multiply (-1)

$$x_1 + x_2 \ge 5$$
$$-x_1 - x_2 \le -5$$

Procedure -

Step-I : Convert the LPP into standard form -

Step-II : Find out the initial basic solution of the problem and construct a initial simplex table.

C_j - - - - - -

C_B	B	X_B	X_1	X_2	X_3	X_n
0	S_1	b_1	a_{11}	a_{12}	a_{13}	a_{1n}
0	S_2	b_2	a_{21}	a_{22}	a_{23}	a_{2n}
⋮	⋮	⋮	⋮			
0	S_n	b_n	a_{m1}	a_{m2}	a_{m3}	a_{mn}
	$Z_j = \sum C_B X_{ij}$ →					
	$Z_j - C_j$ →					

Where B = Basic Variable → (slack variable)

C_B = Coeff. of basic variable

C_j = Coefficient of variable in an objective function

X_B = Value of initial basic solution (R.H.S value)

X_1, X_2, X_3 = Decission Variables

$a_{11}, a_{12} \ldots a_{mn}$ = Coefficient of decision variables of the constraints

Z_j = Evaluation of the value of the decision variable to satisfy the objective fun

$$\boxed{Z_j = \sum_{i=1}^{n} C_B X_{ij}}$$

Step-III : After find out the "$Z_j - C_j$" value check that all the $(Z_j - C_j)$ value are ≥ 0 or not.

(i) If all $(Z_j - C_j)$ value is ≥ 0, then the soln is optimal solution and the value of X_B is the optimal value for the problem.

(ii) If $(Z_j - C_j) \leq 0$, then the solution is not optimal. Go for further steps.

optimality test, Max → $\boxed{Z_j - C_j \geq 0}$

Min → $\boxed{Z_j - C_j \leq 0}$

~~Step-IV : Identify the most negative $(Z_j - C_j)$ value and also the corresponding column.~~

Step-IV : Identify the column which is having the $(Z_j - C_j)$ as negative.

→ If more than one value are negative then select the most negative value and if 2 negative values are having same values, consider any one of the most negative value.

Step-V : Mark an arrow mark on the column which is having the most negative value.

The column is known as "Key column" or "pivot column".

The decission variable corresponding to the pivot column is known as the "Entering Variable".

Step-VI: Now findout the leaving row or key row by using -

$$\boxed{\text{min Ratio} = \frac{x_B}{x_K}, \ \cancel{x} > 0}$$

$$\text{min Ratio} = \frac{\text{Initial basic solution}}{\text{Key column element}}$$

(Min Ratio should be minimum +ve, (0, -, ∞ value will not be consider)

→ From the ratio, find out the least positive value and mark an arrow corresponding to leaving row or key row.

Step-VII: The intersection element of the key row and key column is known as "key element". Convert the key element into unity and the other element of the key column as zero.

" If it is not in the form of 1 devide the same element in the corresponding row.

Step-VIII: To making zero ~~for constructing a new table~~ we have to apply a formula for constructing a new table, having new element value

$$\text{New element} = \text{old Element} - \left[\frac{\text{Key row} \times \text{Key Column element}}{\text{Key Element}}\right]$$

$$\text{Column Ratio for all Column} = \frac{\text{Key Column}}{\text{Key element}}$$

Step - IX : Calculate the value of Z_j and C_j and go to the state and check wheather $Z_j - C_j \geq 0$.

Ex Consider the L.P.P -

$$\text{Maximize } Z = 3x_1 + 2x_2$$

$$\text{s.t } \quad x_1 + x_2 \leq 4$$

$$x_1 - x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Solve by simplex Method.

Sol^n

Step-1

Check whether all the right side Constants are non-Negative.

In this example, all the R.H.S Values are Non Negative.

Convert Canonical to standard form.

$$x_1 + x_2 + S_1 = 4$$

$$x_1 - x_2 + S_2 = 2$$

Obtained L.P.P $\quad \text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2$

$$\text{s.t } \quad x_1 + x_2 + S_1 = 4$$

$$x_1 - x_2 + S_2 = 2$$

$$x_1, x_2, S_1, S_2 \geq 0$$

Step-2

Draw the initial Simplex table,

		C_j	3	2	0	0	
C_B	B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B/X_K
0	S_1	4	1	1	1	0	$4/1 = 4$
← 0	S_2	2	$\boxed{1}$	-1	0	1	$2/1 = 1$
	$Z_j = \sum C_B X_{ij}$	0	0	0	0	0	
	$Z_j - C_j$		↑ -3	-2	0	0	

Key Row / leaving Row

Key Column

Key element

Step-III optimality test — $z_j - C_j \leq 0$
Having two negatives so Solution is not optimal.

Step-IV Identify the Column having most negative value.

Step-V Mark an arrow mark — The Column is known as "Key Column" or "pivot Column".

Step-VI Now findout the leaving row by Min - Ratio

$$\text{Min Ratio} = \frac{X_B}{X_K}$$

Key element must be unity and other elements have to Convert zero.

$$NE = O.E - \left[\frac{\text{Key row} \times \text{key Column}}{\text{Key element}}\right]$$

$$\text{Common Ratio} = \frac{\text{Key Column}}{\text{Key element}} = \frac{\text{Element to be zero}}{\text{Key element}} = 1/(2 \times 1)$$

$X_B = 4-$

First iteration -

	C_j		3	2	0	0	
C_B	B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B/X_K
0	S_1	2	0	$\boxed{2}$	1	-1	$2/2 = 1$
3	X_1	2	1	-1	0	1	$2/-1 = -2$
	$Z_j = \sum C_B x_{ij}$	6	3	-3	0	3	
	$Z_j - C_j$		0	-5 ↑	0	3	

$X_B = 4 - \left[\frac{2 \times 1}{1}\right] = 2$

$X_1 = 1 - \left[\frac{1 \times 1}{1}\right] = 0$

$X_2 = 1 - \left[\frac{-1 \times (-1)}{1}\right] = 2$

$S_1 = 1 - \left[\frac{0 \times 1}{1}\right] = 1$

$S_2 = 0 - \left[\frac{1 \times 1}{1}\right] = -1$

Again you got Negative in $(z_j - C_j)$ i.e. -5. Not optimal.

X_2 is Key Column. Variable Entering Variable.

Again draw the new table.

Second Iteration -

		C_j	3	2	0	0
C_B	B	X_B	X_1	X_2	S_1	S_2
2	x_2	1	0	1	$1/2$	$-1/2$
3	x_1	3	1	0	$1/2$	$1/2$
	$Z_j = \sum C_B X_{Bj}$	11	3	2	$5/2$	$1/2$
	$Z_j - C_j$		0	0	$5/2$	$1/2$ → All +ve.

$$N.E - X_B = O.E - \left[\frac{K.R \times K.C}{K.E}\right]$$

$$= 2 - \left[\frac{2 \times 1}{2}\right] = 3$$

$$X_1 = 1 - \left[\frac{0 \times (-1)}{2}\right] = 1$$

$$X_2 = -1 - \left[\frac{2 \times (-1)}{2}\right] = 0$$

$$S_1 = 0 - \left[\frac{1 \times (-1)}{2}\right] = 1/2$$

$$S_2 = 1 - \left[\frac{-1 \times (-1)}{2}\right] = 1/2$$

All the $Z_j - C_j \geq 0$, Hence the optimal solution is,

$$x_1 = 3 \quad \text{and} \quad x_2 = 1$$

$$\boxed{\text{Max } Z = 11}$$

Ex –

$$\text{Min } Z = x_1 - 3x_2 + 2x_3, \text{ s.t}$$

$$3x_1 - x_2 + 3x_3 \le 7$$

$$-2x_1 + 4x_2 \le 12$$

$$-4x_1 + 3x_2 + 8x_3 \le 10$$

$$x_1, x_2, x_3 \ge 0$$

Sol^n^

$$-(\text{Min } Z) = -(x_1 - 3x_2 + 2x_3)$$

$$\text{Max } Z = -x_1 + 3x_2 - 2x_3 + 0S_1 + 0S_2 + 0S_3$$

$$3x_1 - x_2 + 3x_3 + S_1 = 7$$

$$-2x_1 + 4x_2 + S_2 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$$

		C_j	-1	3	-2	0	0	0	
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	Min Ratio X_B/X_K
0	S_1	7	3	-1	3	1	0	0	-
0	S_2	12	-2	$\boxed{4}$	0	0	1	0	12/4 = 3
0	S_3	10	-4	3	8	0	0	1	10/3 = 3.3
	Z_j	0	0	0	0	0	0	0	
	$Z_j - C_j$	→	1	↑ -3	2	0	0	0	
0	S_1	10	$\boxed{5/2}$	0	3	1	1/4	0	$\frac{10}{5/2}$ = 4
3	x_2	3	-1/2	1	0	0	1/4	0	–
0	S_3	1	-5/2	0	8	0	-3/4	1	–
	Z_j →	9	-3/2	3	0	0	3/4	0	
	$Z_j - C_j$	→	↑ -1/2	0	2	0	3/4	0	
-1	x_1	4	1	0	6/5	2/5	1/10	0	
3	x_2	5	0	1	3/5	1/5	3/10	0	
0	S_3	11	0	0	11	1	-1/2	1	
	Z_j →	11	-1	3	3/5	1/5	8/10	0	
	$Z_j - C_j$	→	0	0	13/5	1/5	8/10	0	

N.E Iteration-1

1st Row $\Rightarrow X_B = 7 - [12 \times -1/4] = 10$

Common Ratio $= -1/4$

$x_1 = 3 - [-2 \times (-1/4)] = 5/2$

$x_2 = -1 - [4 \times (-1/4)] = 0$

$x_3 = 3 - [0 \times (-1/4)] = 3$

$s_1 = 1 - [0 \times (-1/4)] = 1$

$s_2 = 0 - [1 \times (-1/4)] = 1/4$

$s_3 = 0 - [0 \times (-1/4)] = 0$

Iteration-2

(N.E) 2nd row $X_B = 3 - [10 \times -1/5]$
$= 5$

Common Ratio $= \frac{x_C}{x_E} = \frac{-5/2}{5/2} = -1/5$

The optimal solution is $x_1 = 4, x_2 = 5, x_3 = 0$
Min $z = -11$

(Q) Minimize $z = x_1 - 3x_2 + 3x_3$

s.t $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 + 4x_2 \ge -12 \rightarrow -2x_1 - 4x_2 \le 12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

and $x_1, x_2, x_3 \ge 0$

Min $z = -$Max$(-z)$

$-(\text{Min } z) = -(x_1 - 3x_2 + 3x_3)$

Topic:

Artificial Variable Technique

Artificial Variable Technique

→ Artificial variable is added to get the initial feasible solution when the constraint equations are of '=' type or "≥" type.

→ It is not having any physical meaning this technique is used to get the starting basic feasible solution.

$\geq \cdot = -S + A$ → Identity Matrix

Body Matrix + Unit Matrix.

To solve that type of LPP. There are 2 Methods -

1. Big-M or penalty Method
2. Two phase Simplex Method

Big-M Method -

In Big M method the coefficient of artificial variable is term of Capital 'M' which will act a large penalty to artificial variable so it is called penalty method and artificial coefficient is M for that it is also called Big-M Method.

For Max coefficient of A → −M
Min " " A → +M

* Always enter slack variable and artificial variable in the basis, Surplus will not enter

Procedure -

(1) Convert the LPP into standard form by Converting objective function into maximize type.

(2) Add a non-Negative artificial variable to the Constraints having "$\geq$" or "=" type.

(3) put the co-efficient of artificial variable as "$-M$" in the objective function, if it is Maximise type and $+M$ if it is Minimise type.

(4) Solve the LPP by Simplex Method until it satisfies one of the following condition.

(a) If all the $Z_j - C_j$ values are positive or $Z_j - C_j \geq 0$ and no artificial variables are present in the simplex table then the solution is "optimal solution".

(b) If all the $Z_j - C_j$ values are positive and at least one artificial variable present in the simplex table having zero value, then the solution is called "degenerated solution".

(c) If all the $Z_j - C_j$ values are +ve and any artificial variable present in the simplex table is having some (+ve) value, then the solution is "psedo optimal solution".

That type of solution is not having any feasible solution.

(5) Write the model summary of the problem.

(Q) Solve by Big-M Method.

$$\text{Max } Z = 3x_1 + 2x_2$$

s.t. $2x_1 + x_2 \le 2$

$3x_1 + 4x_2 \ge 12$

$x_1, x_2 \ge 0$

Sol^n

$$\text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$$

$2x_1 + x_2 + S_1 = 2$

$3x_1 + 4x_2 - S_2 + A_1 = 12$

$x_1, x_2, S_1, S_2, A \ge 0$

		C_j	3	2	0	0	-M	
C_B	B	X_B	x_1	x_2	S_1	S_2	A	Min Ratio X_B / X_K
← 0	S_1	2	2	$\boxed{1}$	1	0	0	2/1 = 2
-M	A_1	12	3	4	0	-1	1	12/4 = 3
	Z_j →		→ -3M	-4M	0	M	-M	
	$Z_j - C_j$		→ -3M-3	↑ -4M-2	0	M	0	
2	x_2	2	2	1	1	0	0	
-M	A_1	4	-5	0	-4	-1	1	
	Z_j		→ 4+5M	2	2+4M	M	-M	$[2*(\frac{4}{1})] = 4$
	$Z_j - C_j$		→ 1+5M	0	2+4M	M	0	→ All +ve

As artificial variable is having some value in the final simplex table, so the solution is pseudo optimal solution.

Note: While applying simplex Method, whenever an artificial variable happens to leave the basis we delet the artificial variable column from simplex table.

(Q) Use penalty to Solve the following LPP.

Minimize $Z = 5x + 3y$

s.t.
$$2x + 4y \le 12$$
$$2x + 2y = 10$$
$$5x + 2y \ge 10$$
$$x, y \ge 0$$

Soln

$$\text{Min } Z = -\text{Max}(-Z)$$

$$\text{Max } Z = -5x - 3y + 0 \cdot S_1 + 0S_2 - MA_1 - MA_2$$
$$2x + 4y + S_1 = 12$$
$$2x + 2y + A_1 = 10$$
$$5x + 2y - S_2 + A_2 = 10$$
$$x, y, S_1, S_2, A_1, A_2 \ge 0$$

		C_j	-5	-3	0	0	$-M$	$-M$	
C_B	B	X_B	x_1	x_2 y	S_1	S_2	A_1	A_2	Min Ratio X_B/X_K
0	S_1	12	2	4	1	0	0	0	$12/2 = 6$
$-M$	A_1	10	2	2	0	0	1	0	$10/2 = 5$
← $-M$	A_2	10	$\boxed{5}$	2	0	-1	0	1	$10/5 = 2$
	Z_j	$20\cancel{M}$	$-7M$	$-4M$	0	M	$-M$	$-M$	
	$Z_j - C_j$		↑ $-7M+5$	$-4M+3$	0	M	0	0	
← 0	S_1	8	0	$\boxed{16/5}$	1	$2/5$	$\cancel{2/5}$ 0	X	$8/\frac{16}{5} = 5/2$
$-M$	A_1	6	0	$6/5$	0	$2/5$	1	$\cancel{-2/5}$ X	$6/\frac{6}{5} = 5$
-5	x_1	2	1	$2/5$	0	$-1/5$	0	$\cancel{1/5}$	$2/\frac{2}{5} = 5$
	Z_j	$-6M-10$	-5	$-\frac{6}{5}M-2$	0	$-\frac{2}{5}M+1$	$-M$	X	
	$Z_j - C_j$		0	↑ $-\frac{6}{5}M+1$	0	$-\frac{2}{5}M+1$	0	X	

		C_j	-5	-3	0	0	$-M$	$-M$	
C_B	B	X_B	x	y	s_1	s_2	A_1	A_2	Min Ratio
-3	y	$5/2$	0	1	$5/16$	$1/8$	0	X	$(5/2)/(1/8) = 20$
$-M$	A_1	3	0	0	$-3/8$	$\boxed{1/4}$	1	X	12
-5	x	1	1	0	$-1/8$	$-1/4$	0	X	
	Z_j	$-3M - \frac{25}{2}$	-5	-3	$+\frac{3}{8}M - \frac{5}{16}$	$-\frac{1}{4}M + \frac{7}{8}$	$-M$	X	
	$Z_j - C_j$		0	0	$+\frac{3}{8}M$	$-\frac{1}{4}M + 7/8$	0	X	

-3	y	1	0	1	$1/2$	0	X	X
0	s_2	12	0	0	$-3/2$	1	X	X
-5	x	4	1	0	$-1/2$	0	X	X
	Z_j	-23	-5	-3	1	0	X	X
	$Z_j - C_j$		0	0	1	0	X	X

$x = 4$ & $y = 1$ Max $Z = -23$

Min $Z = 23$

2 Phase

(a) Use Big-M Method to solve the following L.P.P –

$$\text{Max } Z = 2x_1 + x_2 + 3x_3$$

$$\text{s.t. } x_1 + x_2 + 2x_3 \le 5$$

$$2x_1 + 3x_2 + 4x_3 = 12$$

$$x_1, x_2, x_3 \ge 0$$

$(x_1 = 3, x_2 = 2, x_3 = 0$

Max Z 8$)$

Topic:

Two Phase Simplex Method

Two Phase Simplex Method

It is another method to solve the given L.P.P involving some artificial variable. The solution can be obtained in two phases.

Phase-I

In this phase we construct an auxiliary LPP leading to the final simplex table containing a basic feasible solution to the original problem.

Step-I

Prepare a auxiliary L.P.P i.e represented by Z^* where all the coefficient are zero and artificial variable coefficient is -1 (Max) and 1 (Min).

New objective function is -

$$Z^* = -A_1, -A_2, -A_3 \ldots\ldots$$

Where A_i is the artificial variable.

Step-II

Write down the auxiliary LPP ~~by simplex method~~ in which the new objective function is to be maximized subjected to given set of constraint equation.

$$Z^* = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A_1 - A_2$$

Step-III

Solve the auxiliary L.P.P by simplex method untill the following cases arises.

Cases

① Max $Z^* < 0$ and at least one artificial variable at +ve value.

② Max $Z^* = 0$, One of the / at least one artificial variable appears zero.

③ Max $Z^* \geq 0$, and no artificial variable present in the basis.

→ If Case - I is arise no need to go for phase - II.
→ If Case ② & ③ arise go for phase - II.

Phase - II

Step - I –
Take the final step of phase - I as a starting solution for the original LPP.

Step - II –
Assign actual cost to the variable in the objective function and zero cost to every artificial variable in the basis.

Step - III
If any artificial variable leaves in phase - 2 delete the artificial variable column.

Step - IV
Apply simplex Method to the modified simplex table obtained at the end of phase - I till an optimal basic feasible solution is obtained.

(Q) Use Two-phase simplex Method to solve,

Max $Z = 5x_1 + 3x_2$

S.t $2x_1 + x_2 \le 1$

$x_1 + 4x_2 \ge 6$

$x_1, x_2 \ge 0$

Sol^n phase-I

~~We Conver~~ Max $Z^* = 0x_1 + 0x_2 + 0S_1 + 0S_2 - 1A_1$

$2x_1 + x_2 + S_1 = 1$

$x_1 + 4x_2 - S_2 + A_1 = 6$

~~phase~~

		C_j	0	0	0	0	-1	
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1	Min Ratio X_B/X_K
0	S_1	1	2	$\boxed{1}$	1	0	0	$1/1 = 1 \to$ Min
-1	A_1	6	1	4	0	-1	1	$6/4 = 1.5$
	Z_j	-6	-1	-4	0	1	-1	
	$Z_j - C_j$		-1	$\uparrow -4$	0	1	0	
0	x_2	1	2	1	1	0	0	
-1	A_1	2	-7	0	-4	-1	1	
	Z_j	-2	+7	0	4	1	-1	
	$Z_j - C_j$		7	0	4	1	0	

All the $Z_j - C_j \ge 0$.

But Max $Z^* < 0$ and one of the artificial variable present at a +ve level, the original LPP doesnot posses any feasible solution. No need to go to phase-2.

(Q) Solve by two-phase simplex Method -

Maximize $Z = 4x_1 - 3x_2 - 9x_3$

s.t $2x_1 + 4x_2 + 6x_3 \geq 15$

$6x_1 + x_2 + 6x_3 \geq 12$

$x_1, x_2, x_3 \geq 0$

Soln Phase - I

$\text{Max } Z^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1.A_1 - 1.A_2$

$2x_1 + 4x_2 + 6x_3 - S_1 + A_1 = 15$

$6x_1 + x_2 + 6x_3 - S_2 + A_2 = 12$

		C_j	0	0	0	0	0	-1	-1	
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Min Ratio X_B/X_K
-1	A_1	15	2	4	6	-1	0	1	0	15/6 = .
-1	A_2	12	6	1	$\boxed{6}$	0	-1	0	1	12/6 = 2
	Z_j	-27	-8	-5	-12	1	1	-1	-1	
	$Z_j - C_j$		-8	-5	↑-12	1	1	0	0	
-1	A_1	3	-4	$\boxed{3}$	0	-1	1	1	-1	3/3 = 1
0	x_3	2	1	1/6	1	0	-1/6	0	1/6	2/(1/6) = 12
	Z_j	-3	4	-3	0	1	-1	0	1	
	$Z_j - C_j$		4	↑-3	0	1	-1	1	0	
0	x_2	1	-4/3	1	0	-1/3	1/3	1/3	-1/3	
0	x_3	11/6	22/18	0	1	1/18	-4/18	-1/18	4/18	
	Z_j	0	0	0	0	0	0	0	0	
	$Z_j - C_j$		0	0	0	0	0	1	1	

All the $Z_j - C_j \geq 0$

Since Max $Z^* = 0$ and no artificial variable present in the basis we go for phase - II

$\text{Max } Z = -4x_1 - 3x_2 - 9x_3 + 0S_1 + 0S_2 - A_1 - A_2$

Phase - II

Consider the final simplex table of Phase I. Also consider the actual cost associated with the original variables.

Delete the artificial variables A_1, A_2 column from the table as these variables are eliminated from the basis in phase-I.

C_B	B	C_j / X_B	x_1 (−4)	x_2 (−3)	x_3 (−9)	S_1 (0)	S_2 (0)	A_1 (−1)	A_2 (−1)	Min Ratio X_B/X_1
−3	x_2	1	$-4/3$	1	0	$-1/3$	$1/3$	X	X	—
−9	x_3	$11/6$	$\boxed{22/18}$	0	1	$1/18$	$-4/18$	X	X	$3/2$
Z_j		$\frac{-39}{2}$	−7	−3	−9	$1/2$	1	X	X	
$Z_j - C_j$			↑ −3	0	0	$1/2$	1	X	X	
−3	x_2	3	0	1	$\frac{12}{11}$	$-3/11$	$+1/11$	X	X	
−4	x_1	$3/2$	1	0	$18/22$	$1/22$	$-4/22$	X	X	
Z_j		−15	−4	−3	$\frac{-72}{11}$	$7/11$	$5/11$	X	X	
$Z_j - C_j$			0	0	$\frac{27}{11}$	$\frac{7}{11}$	$5/11$	X	X	

All $Z_j - C_j \geq 0$

The optimal solution is given by

$$\text{Max } Z = -15$$

$$x_1 = 3/2 \quad x_2 = 3, \quad x_3 = 0$$

Topic:

Degeneracy

Degeneracy – (Tie)

The phenomenon of obtaining a degenerate basic feasible solution in a LPP is known as degeneracy.

→ Degeneracy in LPP may arise
(i) At the initial stage
(ii) At any subsequent iteration stage.

Methods

Step-1 : First find out the rows for which the minimum non-Negative ratio is the same (tie).

Step-2 : Now rearrange the columns of the usual simplex table so that the columns forming the original unit matrix come first in proper order.
→ Sequencing of the elements of the simplex table are $C_B, B, X_B, S_1, S_2, x_1, x_2$.

Step-3 : Find the minimum of the ratio –

$$\left(\frac{\text{Elements of the first column of the unit matrix}}{\text{Corresponding elements of key column}}\right)$$

→ only for the tied rows i.e for the first and 3rd rows.

$$\min\left(\frac{S_1}{\text{Key Col.}}\right)$$

Note

1. 0 values are allowed.
Coeff. of x_1 & x_2 = Body Matrix
S_1 & S_2 = Unit Matrix

(i) If the third row has the min. Ratio then this row will be the key row and the key element can be determined by intersecting the key row with key column. If this is also not unique go to next (ii) of this minimum step.

Step-4 : Now find the minimum of the ratios for the tied rows. If this minimum ratio is unique for the first row, then this row will be the key row for determining the key element

$$\left(\frac{\text{Elements of the second column of unit Matrix}}{\text{Corresponding elements of key column}}\right)$$

If this min. is also not unique go to next step.

Step-5 :

$$\left(\frac{\text{Elements of the third column of the unit matrix}}{\text{Corresponding elements of key column}}\right)$$

(Q) Maximize $z = 3x_1 + 9x_2$

s.t $x_1 + 4x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

Soln

Max $z = 3x_1 + 9x_2 + 0S_1 + 0S_2$

st $x_1 + 4x_1 + S_1 = 8$

$x_1 + 2x_2 + S_2 = 4$

$x_1, x_2, S_1, S_2 \ge 0$

		C_i	3	9	0	0	
C_B	B	X_B	x_1	x_2	S_1	S_2	Min $\frac{X_B}{x_k}$
0	S_1	8	1	4	1	0	$8/4 = 2$
0	S_2	4	1	2	0	1	$4/2 = 2$
	Z_i	0	0	0	0	0	
	$Z_i - C_i$		-3	↑-9	0	0	

$\left.\begin{matrix} 8/4 = 2 \\ 4/2 = 2 \end{matrix}\right\}$ tie

Rearrange the columns of the simplex table so that the initial identity matrix appears -

		C_i	0	0	3	9	
C_B	B	X_B	S_1	S_2	x_1	x_2	Min $\frac{S_1}{x_2}$ (key Column)
0	S_1	8	1	0	1	4	$1/4$
0	S_2 ←	4	0	1	1	$\boxed{2}$	$0/2 = 0$
	Z_i	0	0	0	0	0	
	$Z_i - C_i$		0	0	-3	↑-9	

In degeneracy '0' values are considered to break the tie.

Using Step-3 of the procedure - Min

Hence, S_2 leaves the basis and the key element is 2.

		C_i	0	0	3	9
C_B	B	X_B	S_1	S_2	x_1	x_2
0	S_1	0	1	-2	-1	0
9	x_2	2	0	$1/2$	$1/2$	1
	Z_i	18	0	$9/2$	$9/2$	9
	$Z_i - C_i$		0	$9/2$	$3/2$	0

$C.R = 4/2 = 2$

$N.E = O.E - [\text{Row Element} \times C.R]$

Since all $Z_i - C_i \geq 0$, the soln is optimum.

The optimal solution is $x_1 = 0$, $x_2 = 2$, Max $Z = 18$

(Q) Maximize $Z = 2x_1 + x_2$

s.t. $4x_1 + 3x_2 \le 12$

$4x_1 + x_2 \le 8$

$4x_1 - x_2 \le 8$

$x_1, x_2 \ge 0$

Soln

Maximize $Z = 2x_1 + x_2 + 0S_1 + 0S_2 + 0S_3$

$4x_1 + 3x_2 + S_1 = 12$

$4x_1 + x_2 + S_2 = 8$

$4x_1 - x_2 + S_3 = 8$

		C_j	2	1	0	0	0	
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	Min $\frac{x_B}{x_k}$
0	S_1	12	4	3	1	0	0	12/4 = 3
0	S_2	8	4	1	0	1	0	8/4 = 2
0	S_3	8	4	-1	0	0	1	8/4 = 2
Z_j		0	0	0	0	0	0	
$Z_j - C_j$			↑-2	-1	0	0	0	

tie

Since the min. Ratio is same for 2nd and 3rd rows, it is an indication of degeneracy. Rearrange the column in such a way that the identity Matrix comes first.

		C_j	0	0	0	2	1		
C_B	B	X_B	S_1	S_2	S_3	x_1	x_2	Min S_1/x_1	Min
0	S_1	12	1	0	0	4	3	—	—
0	S_2	8	0	1	0	4	1	0/4	1/4
← 0	S_3	8	0	0	1	$\boxed{4}$	-1	0/4	0/4
Z_j		0	0	0	0	0	0		
$Z_j - C_j$			0	0	0	↑-2	-1		

		C_j	0	0	0	2	1	
C_B	B	X_B	S_1	S_2	S_3	x_1	x_2	Min $\frac{x_B}{x_K}$
0	S_1	4	1	0	-1	0	4	$4/4 = 1$
0	S_2	0	0	1	-1	0	$\boxed{2}$	$0/2 = 0$
2	x_1	2	0	0	$1/4$	1	$-1/4$	—
	Z_j	4	0	0	$1/2$	2	$-1/2$	
	$Z_j - C_j$		0	0	$1/2$	0	$\uparrow -3/2$	
0	S_1	4	1	-2	$\boxed{1}$	0	0	$4/1 = 4$
1	x_2	0	0	$1/2$	$-1/2$	0	1	
2	x_1	2	0	$1/8$	$1/8$	1	0	$\frac{2}{1/8} = 16$
	Z_j	4	0	$3/4$	$-1/4$	2	1	
	$Z_j - C_j$		0	$3/4$	$\uparrow -1/4$	0	0	
0	S_3	4	1	-2	1	0	0	
1	x_2	2	$1/2$	$-1/2$	0	0	1	
2	x_1	$3/2$	$-1/8$	$3/8$	0	1	0	
	Z_j	5	$1/4$	$1/4$	0	2	1	
	$Z_j - C_j$		$1/4$	$1/4$	0	0	0	

Max $z = 5$

$x_1 = 3/2$ & $x_2 = 2$.

Unbounded Solution:-

(1) Maximize $Z = 2x_1 + x_2$

s.t $x_1 - x_2 \le 10$

$2x_1 - x_2 \le 40$

$x_1, x_2 \ge 0$

Sol) $\text{Max } Z = 2x_1 + x_2 + 0S_1 + 0S_2$

$x_1 - x_2 + S_1 = 10$

$2x_1 - x_2 + S_2 = 40$

$x_1, x_2, S_1, S_2 \ge 0$

C_B	B	X_B	x_1 (2)	x_2 (1)	S_1 (0)	S_2 (0)	Min Ratio $\left(\frac{X_B}{x_i}\right)$
0	S_1	10	$\boxed{1}$	-1	1	0	$10/1 = 10$
0	S_2	40	2	-1	0	1	$40/2 = 20$
Z_j		0	0	0	0	0	
$Z_j - C_j$			↑ -2	-1	0	0	
2	x_1	10	1	-1	1	0	—
0	S_2	20	0	$\boxed{1}$	-2	1	20
Z_j		20	2	-2	2	0	
$Z_j - C_j$			0	↑ -3	2	0	
2	x_1	30	1	0	-1	1	
1	x_2	20	0	1	-2	1	
Z_j		80	2	1	-4	3	
$Z_j - C_j$			0	0	↑ -4	3	

$1 - [-1 \times -2]$

Since $Z_j - C_j \le 0$, the solution is not optimum. But all the values in key column are negative which indicates unbounded solution.

→ In some LPP, the solution space becomes unbounded, so that the value of O.F also can be increased indefinitely without a limit

Unbounded feasible region but bounded optimal solution:-

(Q) Maximize $Z = 6x_1 - 2x_2$

s.t $2x_1 - x_2 \le 2$

$x_1 \le 4$

$x_1, x_2 \ge 0$

Sol) $\text{Max } z = 6x_1 - 2x_2 + 0s_1 + 0s_2$

s.t $2x_1 - x_2 + s_1 = 2$

$x_1 + s_2 = 4$

$x_1, x_2 \ge 0$

C_B	B	X_B	x_1 (6)	x_2 (-2)	s_1 (0)	s_2 (0)	Min $\frac{x_B}{x_k}$
0	s_1	2	$\boxed{2}$	-1	1	0	2/2 = 1
0	s_2	4	1	0	0	1	4/1 = 4
Z_j		0	0	0	0	0	
$Z_j - C_j$			↑-6	2	0	0	
6	x_1	1	1	$-1/2$	$1/2$	0	—
0	s_2	3	0	$\boxed{+1/2}$	$-1/2$	1	$\frac{3}{1/2} = 6$
Z_j		6	6	-3	3	0	
$Z_j - C_j$			0	↑-1	3	0	
6	x_1	4	1	0	0	1	
-2	x_2	6	0	1	-1	2	
Z_j		12	6	-2	2	2	
$Z_j - C_j$			0	0	2	2	

$4 - [2 \times 1/2]$

$-(1 \times 1/2)$

-1

-1

$3 - [1 \times$

$\frac{1}{2} - [-1/2 \times -1)$

$(Z_j - C_j) \ge 0$, the solution is optimal

$x_1 = 4, x_2 = 4$ and $\text{Max } z = 12$

The elements of x_2 are negative or zero $(-1, \le 0)$

So the feasible region is not bounded.

* So In a problem there may have unbounded feasible region but still the optimal soln is bounded.

Basic Feasible Solution (BFS)

→ Consider a LPP in the 'Standard form' that contains 'n' variables and 'm' constraints.

→ Assume $n \geq m$

→ A basic solution is obtained by setting $(n-m)$ variables (i.e., Non-basic Variables) equal to zero and solving for the remaining 'm' variables (i.e., Basic variables)

→ No. of basic solution $= {}^{n}C_{m} = \dfrac{n!}{m!\,(n-m)!}$

Example: Max $Z = 3x_1 + 2x_2$

S.t. $2x_1 + x_2 \leq 9$

$x_1 + 2x_2 \leq 9$

$x_1, x_2 \geq 0$

Standard form of LPP

Max $Z = 3x_1 + 2x_2$

s.t. $2x_1 + x_2 + S_1 = 9$

$x_1 + 2x_2 + S_2 = 9$

$x_1, x_2, S_1, S_2 \geq 0$

Here, $n = 4$, $m = 2$

∴ No. of non-basic variables (NBV) $= n - m = 4 - 2 = 2$

No. of basic variables (BV) $= m = 2$

No. of basic solution $= {}^{n}C_{m} = {}^{4}C_{2} = \dfrac{4!}{2!\,(4-2)!} = \dfrac{4\times3\times2\times1}{2\times1\times2\times1} = 6$

	Basic Solution NBV	Basic Solution BV	BFS ?
1	$x_1 = x_2 = 0$	$S_1 = 9,\ S_2 = 9$	Yes
2	$x_1 = S_1 = 0$	$x_2 = 9,\ S_2 = -9$	No
3	$x_1 = S_2 = 0$	$x_2 = 4.5,\ S_1 = 4.5$	Yes
4	$x_2 = S_1 = 0$	$x_1 = 4.5,\ S_2 = 4.5$	Yes
5	$x_2 = S_2 = 0$	$x_1 = 9,\ S_1 = -9$	No
6	$S_1 = S_2 = 0$	$x_1 = 3,\ x_2 = 3$	Yes

Non-degenerate and degenerative basic feasible solution :

⇨ A basic feasible solution is non-degenerate if all the basic variables are greater than zero (> 0).

⇨ A basic feasible solution is degenerate if one or more basic variable(s) is/are zero.

Special Cases In Simplex Method Application :

(1) Tie in the choice of 'entering basic variable' :

To break the 'tie' in the choice of entering variable, arbitrary selection of entering basic variable is suggested. However, if there is a tie betn a decision variable and a slack/surplus variable, select the decision variable as entering basic variable.

(2) Tie in the choice of 'leaving basic variable' :

If there occurs a tie in the choice of outgoing basic variable, arbitrary selection of outgoing/leaving basic variable is suggested.

→ When above tie condition arises in the choice of leaving basic variable, then 'degeneracy' occurs and the solution of the LPP becomes degenerate.

(3)

③ Infinite Number of Optimal Solutions (Multiple Optimal Solutions)

In the optimal simplex table, if a non-basic variable has zero co-efficient in the Z-row, there exists an alternate optimal solution. It is because that non-basic variable can enter the basis without changing the value of 'Z', but causing a change in the values of the basic variables. This variable may be a decision variable or slack or surplus variable.

→ From practical stand point, multiple optimal solutions are useful because they allow us to choose among the many alternatives without sacrificing the objective value.

④ Unbounded Solution :-

While solving a linear programming problem by simplex method, if all the elements of the pivot column / key column become negative or zero, then the LPP will have unbounded solution (as there is no minimum non-negative ratio, θ).

⑤ Infeasible Solution :-

In the optimal simplex table, if at least one artificial variable appears in the basis-column at a non-zero level (with +ve value in solution column) and even though the optimality condition is satisfied, it is the indication of non-feasible solution.

→ This means that the resources of the system are not sufficient to meet the expected demands.

→ The final solution to the problem is not optimal since the objective function contains an unknown quantity 'M'. Such a solution satisfies the constraints but does not optimize the objective function and is also called 'pseudo-optimal solution'.

Topic:

Duality In Linear Programming

Duality in Linear Programming

Every LPP (called the 'Primal') is associated with another LPP (called its 'dual'). The dual problem is an LPP defines directly and systematically from the original (i.e., primal LPP). The two problems are so closely related that the optimal solution of one problem automatically yields the optimal solution to the others.

Foremulation of Dual Problems

Preimal Problem		Dual Problem	
Objective function		Objective function	Constraint type
Max.	≤	Min.	≥
Min.	≥	Max.	≤

* Change the objective function of maximization in primal into minimization one in the dual and vice-versa.
* The number of variable in the primal will be no. of constraints in the dual and vice-versa
* The cost co-efficients in the objective function of the primal will be R.H.S constants of the constraint in the dual and vice-versa.

* Dual of a dual is the primal.

* If either the primal or the dual problem has an unbounded solution, then the solution to the other problem is infeasible and vice-versa.
* If both the preimal and the dual preoblems have feasible solutions, then both have optimal solutions and,

$$\boxed{\text{Max } Z = \text{Min } W}$$

* Primal-Dual Relationship

Primal		Dual
1. Maximization Problem	→	Minimization Problem
2. 'n' variables and 'm' constraints	→	'n' constraints and 'm' variables
3. $\left.\begin{matrix}\le\\\ge\end{matrix}\right\}$ type constraints	→	$\left.\begin{matrix}\ge\\\le\end{matrix}\right\}$ type constraints.
4. Objective function co.efficient	→	R.H.S. constants of the constraints
5. R.H.S. constants of the constraints	→	Objective function co.efficients.
6. '=' type ith constraint	→	Unrestricted ith variable.
7. Variable 'x_j' unrestricted	→	jth constraint '=' type.

* Unrestricted (or Unconstrained) Variables

The unrestricted variable is expressed as a difference of two non-negative variables,

Unrestricted variable, $y = y_1 - y_2$; $y_1 \ge 0$, $y_2 \ge 0$

Value of 'y' will be positive, zero or negative depending upon whether y_1 is larger, equal to or smaller than y_2.

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Canonical and Standard Forms of LPP

After formulating the LPP, the next step is to obtain its solution. Before using any analytic method for solution, the problem must be available in a particular form. The following two forms are:

1- **Canonical Form**:- The general LPP can always be put in the following form, called the canonical form:

$$\text{Max } Z = \sum_{j=1}^{n} c_j x_j ,$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad , i = 1, 2, \dots, m$$

$$x_j \ge 0 \quad , \quad j = 1, 2, \dots, n$$

The characteristics of this form are

(a) The objective function is of the Maximization type

(b) All the constraints are of the "$\le$" type, except for the non-negative restrictions.

(c) All the decision variables are non-negative.

* The minimization of a function, $f(x)$ is equivalent to the maximization of the negative expression of this function, $-f(x)$.

Ex:- $\text{Max } Z = -\text{Min}\{-f(x)\}$

Suppose $\text{Min } Z = c_1x_1 + c_2x_2 + c_3x_3$ is equivalent to

$\text{Max } G = -Z = -c_1x_1 - c_2x_2 - c_3x_3$

with $Z = -G$.

2. **Standard Form**:- The general LPP can always be put in the following form, called the standard form:

$$\text{Max (or Min) } Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad , i = 1, 2, \dots, m$$

$$x_j \ge 0, \quad j = 1, 2, \dots, n$$

$$\text{and} \quad b_i \ge 0$$

Characteristics

(a) All variables are non-negative.

(b) The RHS of each constraint is non-negative

(c) All constraints are expressed as equations.

(d) The objective function may be of Max or Min type.

Duality In Linear Programming

→ One LPP associated with another LPP called duality.
Every LPP (called as primal problem), can be transformed into a dual problem.

Importance -

(1) If the primal contains a large no. of constraints and a smaller number of variable, the labour of computation can be reduced by converting it into the dual problem.

(2) The interpretation of the dual variables from the cost point of view, proves extremely useful in making future decission in the activities being programmed.

Formulation of dual Problem -

(1) Check the form of LPP is in Cannonical form or not. If not then convert it to Canonical form.

(2) If the O.F in primal problem is Maximizan type then consider it as minimisation type in dual problem and vice versa.

(3) Consider the no. of Constraint equation in primal problem as the no. of decission variable in dual problem and vice-versa.

(4) Consider the coefficient of decission variable in the O.F of the primal problem as the right hand side value of the constraints in the dual problem and vice versa.

(5) In forming the constraints for the dual,

We consider the transpose of the body Matrix of the primal problem.

i.e. $A^T W \geq C^T$ (In Max^n)

$A^T W \leq C^T$ (In Min^n)

$O.F \rightarrow \text{Min } z' = b^T W$.

(6) Consider the decission variable are non negative in both primal and dual problem.

(7) Write the model summery.

(8) If a Variable in primal is unrestricted in sign, then the corresponding constraint in the dual will be an equation and vice verse.

Let the primal problem be,

primal L.P.P

$$\text{Max } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{s.t} \rightarrow a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$

$$x_1, x_2 \ldots \geq 0$$

Dual L.P.P

$$\text{Min } z = b_1W_1 + b_2W_2 + \cdots + b_mW_n$$

$$\text{s.t} \quad a_{11}W_1 + a_{21}W_2 + \cdots + a_{n1}W_n \geq C_1$$

$$a_{12}W_1 + a_{22}W_2 + \cdots + a_{n2}W_n \geq C_2$$

$$\vdots$$

$$a_{m1}W_1 + a_{m2}W_2 + \cdots + a_{mn}W_n \geq C_n$$

$$W_1, W_2, W_3 \ldots \geq 0$$

Where, $W_1, W_2, W_3 \cdots W_m$ are Called dual variables.

In Matrix form :-

primal form | Dual form

Condⁿ-I

(a) Max $Z = CX$
s.t $AX \leq b$
$X \geq 0$

(a) Min $Z = b^T W$
$A^T W \geq C^T$
$W \geq 0$

(b) Min $Z = CX$
s.t $AX \geq b$
$X \geq 0$

(b) Max $Z = b^T W$
$A^T W \leq C^T$
$W \geq 0$

Condⁿ-II

(a) Max $Z = CX$
$AX = b$
$X \geq 0$

(a) Min $Z = b^T W$
$A^T W \geq C^T$
$\cancel{W \geq 0}$
W is unrestricted
(Means W_1 & W_2)

(b) Min $Z = CX$
$AX = b$
$X \geq 0$

(b) Max $Z = b^T W$
$A^T W \leq C^T$
W is unrestricted (W_1 & W_2)

Condⁿ-III

(a) Max $Z = CX$
$AX = b$
X is unrestricted

(a) Min $Z = b^T W$
$A^T W = C^T$
W is unrestricted
$W_1 \geq 0$, W_2

(b) Min $Z = CX$
$AX = b$
X is unrestricted

(b) Max $Z = b^T W$
$A^T W = C^T$
W is unrestricted.

Note

* Unrestricted Means - Difference b/w two non-negative.

$W = W' - W''$

$W' < W''$, $W' > W''$, $W' = W''$

Q) Write the dual of the following primal L.P.P problem.

$$\text{Max } Z = x_1 + 2x_2 + x_3$$

$$\text{s.t} \quad 2x_1 + x_2 - x_3 \leq 2$$

$$+2x_1 + x_2 - 5x_3 \geq -6 \rightarrow$$

$$4x_1 + x_2 + x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

Soln primal

$$\text{Max } Z = x_1 + 2x_2 + x_3$$

$$\text{s.t} \quad 2x_1 + x_2 - x_3 \leq 2$$

$$\cancel{-2x_1 + x_2 + 5x_3 \leq 6}$$

$$-2x_1 - x_2 + 5x_3 \leq 6$$

$$4x_1 + x_2 + x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

$$\text{Min } Z = b^T w$$

$$\text{s.t} \quad A^T w \geq C^T$$

$$w \geq 0$$

Dual form

$$b = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix} \rightarrow b^T = \begin{bmatrix} 2 & 6 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 5 \\ 4 & 1 & 1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -1 & 1 \\ -1 & 5 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \rightarrow C^T = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let w_1, w_2, w_3 are dual variable.

$$\text{Min } Z = 2w_1 + 6w_2 + 6w_3$$

s.t

$$2w_1 - 2w_2 + 4w_3 \geq 1$$

$$1w_1 - 1w_2 + 1w_3 \geq 2$$

$$-1w_1 + 5w_2 + 1w_3 \geq 1$$

$$w_1, w_2, w_3 \geq 0$$

(Q) Find dual form -

$$\text{Max } Z = 3x_1 - x_2 + x_3$$
$$4x_1 - x_2 \le 8$$
$$8x_1 + x_2 + 3x_3 \ge 12$$
$$5x_1 - 6x_3 \le 13$$
$$x_1, x_2, x_3 \ge 0$$

Sol^n

$$\text{Max } Z = 3x_1 - x_2 + x_3$$
$$4x_1 - x_2 \le 8$$
$$-8x_1 - x_2 - 3x_3 \le -12$$
$$5x_1 - 6x_3 \le 13$$
$$x_1, x_2, x_3 \ge 0$$

$$C = \begin{bmatrix} 3 & -1 & 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -8 & -1 & -3 \\ 5 & 0 & -6 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & -6 \end{bmatrix}$$

$$b = \begin{bmatrix} 8 \\ 12 \\ 13 \end{bmatrix}, \quad b^T = \begin{bmatrix} 8 & -12 & 13 \end{bmatrix}$$

$$\text{Min } Z = b^T W$$
$$\text{Min } Z = 8W_1 - 12W_2 + 13W_3$$
$$A^T W \ge C^T$$
$$4W_1 - 8W_2 + 5W_3 \ge 3$$
$$-W_1 - W_2 + 0W_3 \ge -1$$
$$0W_1 - 3W_2 - 6W_3 \ge 1$$
$$W_1, W_2, W_3 \ge 0$$

(Q) Min $Z = 2x_2 + 5x_3$

s.t $x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0$

Soln Since the given primal problem is not in the Canonical form, we interchange inequality of constraints.

The third constraint is also an eqn and can be converted into two equations.

primal

$$\text{Min } z = 0x_1 + 2x_2 + 5x_3$$

S.T $x_1 + x_2 + 0x_3 \geq 2$

$-2x_1 - x_2 - 6x_3 \geq -6$

$x_1 - x_2 + 3x_3 \leq 4$

$x_1 - x_2 + 3x_3 \geq 4$

Again $\text{Min } z = 0x_1 + 2x_2 + 5x_3$

S.T $x_1 + x_2 + 0x_3 \geq 2$

$-2x_1 - x_2 - 6x_3 \geq -6$

$-x_1 + x_2 - 3x_3 \geq -4$

$x_1 - x_2 + 3x_3 \geq 4$

Dual

Let w_1, w_2, w_3', w_3'' are the dual variables.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -6 \\ -1 & 1 & -3 \\ 1 & -1 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & -2 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & -6 & -3 & 3 \end{bmatrix}$$

$$b = \begin{pmatrix} 2 \\ -6 \\ -4 \\ 4 \end{pmatrix}, \quad b^T = \begin{pmatrix} 2 & -6 & -4 & 4 \end{pmatrix}$$

$C = \begin{pmatrix} 0 & 2 & 5 \end{pmatrix}$, $C^T = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$

Since there are four constraints in the primal, we have four dual, w_1, w_2, w_3', w_3''

$$\text{Max } Z = b^T W$$

$$\text{Max } Z = 2w_1 - 6w_2 - 4w_3' + 4w_3''$$

S.T.C $\quad A^T W = C^T$

$$w_1 - 2w_2 - w_3' + w_3'' \leq 0$$
$$w_1 - w_2 + w_3' - w_3'' \leq 2$$
$$0w_1 - 6w_2 - 3w_3' + 3w_3'' \leq 5$$

$$\text{Max } Z = 2w_1 - 6w_2 + 4(w_3'' - w_3')$$

S.T.C
$$w_1 - 2w_2 + (w_3'' - w_3') \leq 0$$
$$w_1 - w_2 - (w_3'' - w_3') \leq 2$$
$$0w_1 - 6w_2 + 3(w_3'' - w_3') \leq 5$$

$$\text{Max } Z = 2w_1 - 6w_2 + 4w_3$$

S.T.C
$$w_1 - 2w_1 + w_3 \leq 0$$
$$w_1 - w_2 - w_3 \leq 2$$
$$0w_1 - 6w_2 + 3w_3 \leq 5$$

$w_1, w_2 \geq 0$, w_3 is unrestricted

ex - Q

Find Dual of following problem –

$$\text{Max } Z = x + 2y$$
$$2x + 3y \geq 4$$
$$3x + 4y = 5$$

$x \geq 0$ and y is unrestricted

Sol^n $\quad \text{Max } Z = x + 2y$

~~$2x + 3y \geq 4$~~

$$-(2x + 3y) \leq -4$$

Cannonical form.

$$\text{Max } Z = x + 2y$$

S.T.C $\quad -(2x+3y) \le -4$

$$-2x - 3y \le -4$$

$$3x + 4y = 5 \;\Big\langle\; \begin{array}{l} 3x + 4y \le 5 \\ 3x + 4y \ge 5 \end{array}$$

Since the variable y is unrestricted, it can be expressed as $y = y' - y''$, $y', y'' \ge 0$

primal

$$\text{Max } Z = x + 2y$$

$$-2x - 3y \le -4$$

$$3x + 4y \le 5$$

$$-3x - 4y \le -5$$

$x \to W_1$

$y \to W_2$

primal

$$\text{Max } Z = x + 2(y' - y'')$$

$$-2x - 3(y' - y'') \le -4$$

$$3x + 4(y' - y'') \le 5$$

$$-3x - 4(y' - y'') \le -5$$

$$C = \begin{pmatrix} 1 & 2 & -2 \end{pmatrix} \qquad C^T = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & -3 & 3 \\ 3 & 4 & -4 \\ -3 & -4 & 4 \end{pmatrix} \qquad A^T = \begin{pmatrix} -2 & 3 & -3 \\ -3 & 4 & -4 \\ 3 & -4 & 4 \end{pmatrix}$$

$$b = \begin{pmatrix} -4 \\ 5 \\ -5 \end{pmatrix} \qquad b^T = \begin{pmatrix} -4 & 5 & -5 \end{pmatrix}$$

Dual

$$\text{Min } Z = -4W_1 + 5W_2' - 5W_2''$$

S.T.C $\quad -2W_1 + 3W_2' - 3W_2'' \ge 1$

$$-3W_1 + 4W_2' - 4W_2'' \ge 2$$

$$3W_1 - 4W_2' + 4W_2'' \ge -2$$

$W_1, W_2', W_2'' \ge 0$

$$\text{Min } Z = -4w_1 + 5(w_2' - w_2'')$$

S.T. $-2w_1 + 3(w_2' - w_2'') \geq 1$

$-3w_1 + 4(w_2' - w_2'') \geq 2$

$3w_1 - 4(w_2' - w_2'') \geq -2$

Now $\text{Min } Z = -4w_1 + 5w_2$

S.T

$-2w_1 + 3w_2 \geq 1$

$-3w_1 + 4w_2 \geq 2$

$3w_1 - 4w_2 \geq -2$

$w_1 \geq 0$ and w_2 is unrestricted

Now Convert equ^n (2) & (3) in to equality

$\Rightarrow$ $\text{Min } Z = -4w_1 + 5w_2$

S.T $-2w_1 + 3w_2 \geq 1$

$-3w_1 + 4w_2 = 2$

$w_1 \geq 0$ and w_2 is unrestricted.

(Q) Min $Z = 4x_1 + 5x_2 - 3x_3$

S.T $x_1 + x_2 + x_3 = 22$

$3x_1 + 5x_2 - 2x_3 \leq 65$

$x_1 - 7x_2 + 4x_3 \geq 120$

$x_1, x_2 \geq 0$ and x_3 is unrestricted.

Topic:

Dual Simplex Method

Dual Simplex Method

(primal is optimal but not feasible)

Algorithm -

(1) Check the objective function of the problem is in maximization form or not, if not convert it into maximization form.

(2) Check the constraint equation are less than or equal to type or not. If not convert it into less than equal to type by multiplying both side by -1.

(3) ~~Express~~ Convert the constraint equation into standard form (less than equal to type to = by adding slack variable)

(4) Calculate the basic solution and construct an initial table.

(5) Check the value of $Z_j - C_j$ -

Case-1: If $(Z_j - C_j) \geq 0$ and all value of X_B are greater than zero than the solution is optimal.

Case-2: If $(Z_j - C_j) \geq 0$ and at least one X_B value is less than zero then proceed for further steps.

Case-3: If any $(Z_j - C_j) < 0$ then the solution is infeasible.

(6) Identify the most negative value of X_B and consider the row as key row.

(7) Find the ratio b/w the values of $Z_j - C_j$ and key row and identify the

least negative value from the ratio. The column corresponding to value is key column.

(8) The intersection point of the key column and key row is known as key element. Do the same process to proceed for the next iteration as per the simplex method.

(9) Continue the process until it satisfy one of the condition of step-5.

* Consider the ratios with negative denominator alone.
* If there is no such ratio with negative denominator, then the problem does not have a feasible solution.

* Leaving Row = Most Negative value of X_B Column

(7) * Entering Column = Min. Ratio of $\left|\frac{Z_i - C_i}{X_{rK}}\right|$, $X_{rK} < 0$

(Only Negative values of X_{rK} considered)

X_{rK} - Elements of leaving Row.

(Q) Use dual Simplex Method to solve the LPP.

$$\text{Max } Z = -3x_1 - x_2$$
$$x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Soln

$$\text{Max } Z = -3x_1 - x_2$$
$$-x_1 - x_2 \leq -1$$
$$-2x_1 - 3x_2 \leq -2$$
$$x_1, x_2 \geq 0$$

Convert the inequality to equality by adding slack variable.

$$\text{Max } Z = -3x_1 - x_2 + 0S_1 + 0S_2$$
$$-x_1 - x_2 + S_1 = -1$$
$$-2x_1 - 3x_2 + S_2 = -2$$
$$x_1, x_2, S_1, S_2 \geq 0$$

An initial basic solution of the modified L.P.P

		C_j	-3	-1	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
0	S_1	-1	-1	-1	1	0
0	S_2	-2 ✓	-2	-3	0	1
	Z_j	0	0	0	0	0
	$Z_j - C_j$		3	↑ 1	0	0

Since all $Z_j - C_j \geq 0$ and all $X_B < 0$, the solution is not optimal.

Since X_{B2} is most negative, the corresponding basic variable S_2 leaves the basis.

Min ~~Max~~ $\left\{ \left| \frac{Z_j - C_j}{X_{rk}} \right| , X_{rk} < 0 \right\}$

Min $\left\{ \left|\frac{3}{-2}\right| , \left|\frac{1}{-3}\right| \right\} = 1.33 , 0.33$

S_2 is leaving & x_2 is entering

first iteration -

	C_j		-3	-1	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
0	S_1	$-1/3$	$-1/3$	0	1	$\boxed{-1/3}$
-1	x_2	$2/3$	$2/3$	1	0	$-1/3$
	Z_j	$-2/3$	$-2/3$	-1	0	$1/3$
	$Z_j - C_j$		$7/3$	0	0	$1/3$

~~N.E = old~~ Key

$$N.E = O.E - [\text{Key Row} \times C.R]$$

$$C.R = \frac{\text{Key Column Element}}{\text{Key Element}} = \frac{-1}{-3} = 1/3$$

$X_B \Rightarrow -1 - [-2/3] = -1 + 2/3$

$X_1 \Rightarrow -1 - [-2 \times 1/3] = -1 + 2/3 =$

$S_1 \Rightarrow 1 - [0 \times 1/3] = 1$

$S_2 \Rightarrow 0 - [1 \times 1/3] = -1/3$

Since all $Z_j - C_j \geq 0$ and $X_{B1} = -1/3 < 0$, the current solution is not optimal.

$X_{B1} = -1/3 < 0$ S_1 leaves the basis.

~~Max $\frac{Z_j - C_j}{a_{i1}}$ = Max $\frac{1}{3} / (-1/3) = -1$~~

Min $\left| \frac{Z_j - C_j}{X_{rk}} \right|$, → Min $\left\{ \left|\frac{7/3}{-1/3}\right| , \left|\frac{1/3}{-1/3}\right| \right\}$

Min = 7, 3

S_1 will leave and S_2 will enter.

second iteration -

	C_j		-3	-1	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2
0	S_2	1	1	0	-3	1
-1	x_2	1	1	1	-1	0
	Z_j	-1	-1	-1	1	0
	$Z_j - C_j$		2	0	1	0

$C R = 1$

$x_B \Rightarrow 2/3 - [-1/3] = 1$

$x_1 =$

$x_2 = 1 - 1[$

$0 - [1 \times 1] = -1$

Since all $Z_j - C_j \geq 0$ and $X_B \geq 0$, an optimal solution reached.

The optimal solution is $x_1 = 0$ and $x_2 = 1$

Maximize $Z = -1$.

Differentce b/w Dual Simplex and Simplex

Simplex Method	Dual Simplex Method
→ Constraint equation should be greater than equal to or $\leq$ type.	→ Constraint equ^n should always $\leq$ type.
→ First we identify key column (entering element) then key rows (leaving variable.	→ first we identify key row (leaving variable) then we identify the key column (Entering variable '
→ The solution is optimal when all $z_i - c_i \geq 0$.	→ All $z_i - c_i \geq 0$ and $x_B \geq 0$.

Advantages.

(Q) Solve by the dual simplex Method -

$$\text{Min } Z = 5x_1 + 6x_2$$
$$\text{s.t. } x_1 + x_2 \ge 2$$
$$4x_1 + x_2 \ge 4$$
$$x_1, x_2 \ge 0$$

Soln

$$\text{Max } Z = -5x_1 - 6x_2$$
$$\text{s.t. } -x_1 - x_2 \le -2$$
$$-4x_1 - x_2 \le -4$$
$$x_1, x_2 \ge 0$$

$$\rightarrow \quad \text{Max } Z = -5x_1 - 6x_2 + 0s_1 + 0s_2$$
$$-x_1 - x_2 + s_1 = -2$$
$$-4x_1 - x_2 + s_2 = -4$$
$$x_1, x_2, s_1, s_2 \ge 0$$

Initial table

		C_j	-5	-6	0	0
C_B	B	X_B	x_1	x_2	s_1	s_2
0	s_1	-2	-1	-1	1	0
0	s_2	-4	$\boxed{-4}$	-1	0	1
	Z_j	0	0	0	0	0
	$Z_j - C_j$		↑5	6	0	0
0	s_1	-1	0	-3/4	1	$\boxed{-1/4}$
-5	x_1	1	1	1/4	0	-1/4
	Z_j	-5	-5	-5/4	0	5/4
	$Z_j - C_j$		0	19/4	0	↑5/4
0	s_2	4	0	3	-4	1
-5	x_1	2	1	1	-1	0
	Z_j	-10	-5	-5	5	0
	$Z_j - C_j$		0	1	5	0

$$\text{Min } \left| \frac{Z_j - C_j}{X_{RC}} \right|$$

$$\text{Min } \left| \frac{5}{-4}, \frac{6}{-1} \right|$$

$$\text{Min } |-1.3|, |-6|$$

$$\text{Min } (1.3)$$

$$CR = 1/4$$

$$-2 - [-4 \times 1/4]$$

$$\rightarrow -2 + 1 = -1$$

$$\rightarrow -1 - (-1 \times 1/4)$$

$$\Rightarrow -1 + 1/4$$

$$0 - [1 \times 1/4]$$

$$-[-3/4]$$

$$0 - [1 \times 1]$$

Topic:

Revised Simplex Method

$Z_j - C_j \geq 0$ and also $X_B \geq 0$ the current feasible solution is optimum, The optimal solution is given by $x_1 = 2$, $x_2 = 0$, Max $Z = -10$

i.e Min $Z = 10$

Revised Simplex Method

Simplex method is not very economical as many calculations are carried out and stored in the memory of the computer.

→ Revised simplex is modification of simplex method.

Procedure –

Step-1: Check the objective function is maximize or minimize. If minimize convert it into maximize and check the constraints are "≥" type or "≤" type. If it is "≥" type convert into "≤" type.

Step-2: Consider the non-basic variable as zero and construct the matrix for basic variable and basic solution from the LPP.

i.e $B X_B = b$

$\Rightarrow X_B = B^{-1} b$

Where, B^{-1} = Inverse matrix of basic variable / Unit Matrix

X_B = Basic Solution

b = Right hand side value of constraint.

Find an initial basic feasible solution and form the auxiliary matrix.

S.t $\boxed{\hat{B} = \begin{pmatrix} B & 1 \\ -C_B & 1 \end{pmatrix}}$ and $\boxed{\hat{B}^{-1}_{curr} = \begin{pmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{pmatrix}}$

Step-3 Considering the objective function $Z = CX$ as an additional constraint. form $\hat{A}$ and $\hat{b}$

Such that $\boxed{\hat{A} = \begin{pmatrix} A \\ -C \end{pmatrix}}$ and $\boxed{\hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}}$

Step-4 Compute the net evaluation -

$$\boxed{Z_i - C_i = \begin{pmatrix} C_B B^{-1} & 1 \end{pmatrix} \begin{pmatrix} A \\ -C \end{pmatrix}}$$

i.e By multiplying the successive column of $\hat{A}$ with the last row of $\hat{B}^{-1}$.

(i) If all $Z_i - C_i \geq 0$, the current basic soln is an optimum solution

(ii) If at least one $Z_i - C_i < 0$, determine the most negative of them and corresponding ~~Varing~~ variable is entering variable or entering column.

Step-5 Compute $\boxed{\hat{X}_k = \hat{B}^{-1}_{curr} \hat{a}_k}$

Case 1: If all $\hat{X}_{ik} \leq 0$, then there exists an unbounded soln.

Case-2: If at least one $X_{ik} > 0$, Consider the current X_m and determine the leaving row by computing $\text{Min}\left\{\frac{X_{Bi}}{X_{ik}}, X_{ik}\right\}$

Go to step 6

Step-6 - Write down the result obtained from step (2) to step (5) in the revised simplex table -

Step-7 - Convert the Key ~~leading~~ element unity and all other elements of the key column to zero.

Step-8: Go to step (4) and repeat the procedure untill an optimum basic feasible solution is obtained.

Ex-1 Use the revised Simplex method to solve the following LPP.

$$\text{Max } Z = 6x_1 - 2x_2 + 3x_3$$

$$\text{s.t } \quad 2x_1 - x_2 + 2x_3 \le 2$$

$$x_1 + 4x_3 \le 4$$

$$x_1, x_2, x_3 \ge 0$$

Soln

$$\text{Max } Z = 6x_1 - 2x_2 + 3x_3 + 0S_1 + 0S_2$$

$$\text{s.t } \quad 2x_1 - x_2 + 2x_3 + S_1 = 2$$

$$x_1 + 4x_3 + S_2 = 4$$

$$x_1, x_2, x_3, S_1, S_2 \ge 0$$

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & S_1 & S_2 \end{array} \\ \begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \end{pmatrix} \end{array} \qquad b = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 6 & -2 & 3 & 0 & 0 \end{pmatrix} \qquad C_B = \begin{pmatrix} 0 & 0 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} A \\ -C \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \\ -6 & 2 & -3 & 0 & 0 \end{pmatrix}$$

$$\hat{B}^{-1}_{curr} = \begin{pmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now findout $Z_i - C_i$ -

$$Z_i - C_i = \left(C_B B^{-1} \; 1\right) \hat{A}$$

$$= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}_{1\times 3} \begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \\ -6 & 2 & -3 & 0 & 0 \end{pmatrix}_{3\times 5}$$

$$= \begin{pmatrix} 0+0-6 & 0+0+2 & 0+0-3 & 0 & 0 \end{pmatrix}_{1\times 5}$$

$$= \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & S_1 & S_2 \end{array} \\ \begin{pmatrix} -6\uparrow & 2 & -3 & 0 & 0 \end{pmatrix} \end{array}$$

Since $Z_i - C_i = -6$ is most negative, x_1 enters the basis.

$$\hat{X}_1 = \hat{B}^{-1}_{curr} \hat{a}_1$$

($\hat{a}_1$ is the 1st Column of $\hat{A}$ Matrix)

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}$$

$\hat{b}_1$ - Right hand side value

$$= \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}$$

$$\hat{X}_B = \hat{B}^{-1}_{curr} \hat{b}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

Find out the leaving variable or leaving row/ key row - The Revised Simplex table is -

X_B	B	B^{-1}_{curr}	$\hat{x}_1$	Min Ratio (X_B/X_k)
2	S_1	1 0 0	$\boxed{2}$	$2/2 = 1$
4	S_2	0 1 0	1	$4/1 = 4$
0		0 0 1	-6	

1st Iteration–

$$B^{-1}_{curr} = \begin{pmatrix} \underbrace{\begin{matrix} 1/2 & 0 \\ -1/2 & 1 \end{matrix}}_{B^{-1}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \underbrace{\begin{matrix} 3 & 0 \end{matrix}}_{C_B B^{-1}} & 1 \end{pmatrix}$$

C·R = 1/2

N·E = 0 − [1 × 1/2]

C·R = −6/2 =

0 − [1 × −3]

Again $(z_j - c_j) = \begin{pmatrix} C_B B^{-1} & 1 \end{pmatrix} \hat{A}$

$$= \begin{pmatrix} 3 & 0 & 1 \end{pmatrix}_{1\times 3} \begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \\ -6 & 2 & -3 & 0 & 0 \end{pmatrix}_{3\times 5}$$

$$= \begin{pmatrix} 6+0-6 & -3+0+2 & 6+0-3 & 3+0+0 & 0+0+0 \end{pmatrix}$$

$$= \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 \end{matrix} \\ \begin{pmatrix} 0 & -1 & 3 & 3 & 0 \end{pmatrix} \end{array}$$

Since $z_j - c_j = -1$ is most Negative, x_2 enters the basis.

$$\hat{x}_2 = \hat{B}^{-1}_{curr} \cdot \hat{a}_2$$

(a_2 is the second column of $\hat{A}$ Matrix)

$$= \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}_{3\times 3} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}_{3\times 1}$$

3×1

$$= \begin{pmatrix} -1/2+0+0 \\ 1/2+0+0 \\ -3+0+2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1/2 \\ -1 \end{pmatrix}$$

$$\hat{X}_B = \hat{B}^{-1}_{curr}\,\hat{b} = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}_{3\times 3} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}_{3\times 1}$$

$$= \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

Revised Simplex table is

$\hat{X}_B$	B	$\hat{B}^{-1}_{curr}$	$\hat{x}_2$	Ratio
1	x_2	1/2 0 0	-1/2	–
3	s_2	-1/2 1 0	$\boxed{1/2}$	6
6		3 0 1	$\uparrow$ -1	

Second iteration –

$$\hat{B}^{-1}_{curr} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ \underbrace{2 \quad 2}_{C_B B^{-1}} & & 0 \end{pmatrix}$$

$-[1/2 \times (-1)]$
$-[-1]^{-1/2}$
$-[-1/2 \times -2]$
$0 -$

$$z_i - c_i = \begin{pmatrix} C_B B^{-1} & 1 \end{pmatrix} \hat{A}$$

$$= \begin{pmatrix} 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \\ -6 & 2 & -3 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4+2-6 & -2+2 & 4+8-3 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 9 & 2 & 2 \end{pmatrix}$$

Since all $z_i - c_i \geq 0$, the current feasible solution is optimal.

$$\hat{X}_B = \hat{B}^{-1}_{curr}\,\hat{b} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix}$$

The optimal Soln is -
1st x_1 enter, then x_2 enters ~~a~~ So
. $x_1 = 4$, $x_2 = 6$, $x_3 = 0$
and Max $z = 12$.

Module-II

Topic:

Transportation Problem

MODULE-II

Transportation Problem :-

Transportation problem is a special Case of LPP whose main objective is to transport homogeneous Commodity from different origin to different destination in such a way that the transportation Cost is maximum.

Objective - To Minimise the transportation Cost

Basic Inputs are -

1. Source / Destination
2. Demand / Supply
3. Unit Cost
4. Quantity of product

Let m = No. of Source

n = No. of Destination

a_i = Quantity of supply

b_j = Quantity of Demand

i = Identity of Source

j = Identity of destination

C_{ij} = Cost of quantity to be transport from Source to destination.

x_{ij} = Quantity of product to be transported from source to destination

Assumption in transportation problem :-

1. The total quantity of the item available at different sources is equal to

total requirement of ~~total~~ at different destination.

(2) Item should be transported conveniently from all sources to destination.

Tabular form:

Destinations

SOURCE	1	2	3	.... n	supply
1	C_{11} x_{11}	C_{12} x_{12}	C_{13} x_{13}	C_{1n} x_{1n}	a_1
2	C_{21} x_{21}	C_{22} x_{22}	C_{23} x_{23}	C_{2n} x_{2n}	a_2
3	C_{31} x_{31}	C_{32} x_{32}	C_{33} x_{33}	C_{3n} x_{3n}	a_3
⋮ m	C_{m1} x_{m1}	C_{m2} x_{m2}	C_{m3} x_{m3}	C_{mn} x_{mn}	a_m
Demand	b_1	b_2	b_3	b_n	

Mathematical form:-

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}\cdot x_{ij}$$

S.T

$$\sum_{j=1}^{n} x_{ij} = a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \geq 0$$

Types of Transport problem:-

(I) Balanced (Σ supply $= \Sigma$ Demand)

(II) Unbalanced (Σ supply $\neq \Sigma$ Demand)

*→ Unbalanced type is converted to balanced type by adding some "Dummy"

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variables :

Eg -

					Supply
					5
					10
					15
Demand	20	10	5	2	

Σ Supply = 30 , Σ Demand = 37

→ If $\Sigma S > \Sigma D$, then one dummy Column is introduced into the balanced condition.

→ If $\Sigma S < \Sigma D$, then one dummy row is introduced to convert into balanced condition.

→ The unit cost of Dummy row/ column is zero.

Solution of Transportation problem :-

The solution of a transportation problem can be obtained in 2 ways -

(i) Initial solution

(ii) Optimal solution

For initial solution there are 3 different Method -

(i) North - West Corner Rule (NWCR)

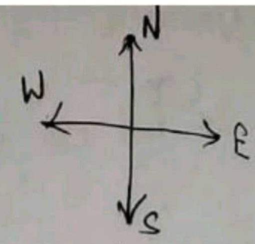
(ii) Least Cost method or Matrix Minima Method.

(iii) Vogel's approximation Method (VAM)

The improved solution of the initial basic solution is called optimal solution, which is the 2nd stage of the solution and can be obtained by MODI (Modified distribution Method)

(i) Initial Solution -

(a) North - west Corner Rule (NWCR)



Steps

(i) Check the transportation problem is balanced or not.

(ii) Start the allocation at the north, west Corner of the transportation table and give the allocation value which is min. along the supply and demand.

(iii) If the supply is fulfilled, make the value as zero and move to next row vertically.

(iv) If the demand is fulfilled, make the value as zero and move to the next column horizontally.

(v) Continue the process and move downward to the South-east Corner until it satisfies the row and column requirement i.e. Total supply and Total Demand = 0.

(Q) Determine the initial basic feasible solution to the following transportation problem using N.W.C.R

Origin/Destination	D_1	D_2	D_3	D_4	Supply
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
Demand	5	8	7	14	

Ans Since $\Sigma S = \Sigma D$, there exists a feasible solution to the transportation problem.
~~we obta~~

	D_1	D_2	D_3	D_4	Supply	
O_1	19 [5]	30 [2]	50 X	10 X	7/1	2/0
O_2	70 X	38 [6]	40 [3]	60 X	9/3/0	
O_3	40 X	8 X	70 [4]	20 [14]	18/14/0	
Demand	5/0	8/6/0	7/4/0	14/0	34	

Total transportation Cost =
$(19\times5)+(30\times2)+(3\times6)+(40\times3)+(70\times4)+(20\times14)$
$= 1015$ /-

(Q) Find the initial basic feasible solution of the transportation problem.

Destination

Origin	D_1	D_2	D_3	D_4	Supply
O_1	6	1	9	3	70
O_2	11	5	2	8	55
O_3	10	12	4	7	70
Demand	85	35	50	45	

Soln $\Sigma D = 215$, $\Sigma S = 195$

$\Sigma D > \Sigma S$

As ΣS is less than Demand we have to add one dummy row whose

$\Sigma D - \Sigma S = 215 - 195 = 20$

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Supply value is 20 and unit cost is zero

Destination

Origin					Supply
	6	1	9	3	70
	11	5	2	8	55
	10	12	4	7	70
	0	0	0	0	
Demand					

Destination

Origin					Supply
	6 [70]	1 ×	9 ×	3 ×	~~70~~/0
	11 [15]	5 [35]	2 [5]	8 ×	~~55~~/~~40~~/~~5~~/0
	10 ×	12 ×	4 [45]	7 [25]	~~70~~/25
	0 ×	0 ×	0 ×	0 [20]	~~20~~/0
Demand	~~85~~/~~15~~/0	~~35~~/0	~~50~~/~~45~~/0	~~45~~/~~20~~/0	

Total transportation Cost $= (6\times 70) + (11\times 15) + (5\times 35) + (2\times 5) + (4\times 45) + (7\times 25) + (0\times 20)$

$= \underline{1125}$

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(II) Least- Cost Method (or) Matrix Minima Method –

(i) Check the transportation problem is balanced or not if not balanced it.

(ii) Choose the cell which is having min. unit Cost.

(iii) Allocate the min. value among the supply and demand to that cell.

(iv) Continue the process until it satisfy the row and column requirement i.e supply & Demand = 0

(v) Write the Model Summary of the Problem.

(Q) Determine the initial basic feasible solution for the following T.P.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	14	5	14
O_2	6	9	1	7	16
O_3	4	1	6	4	5
Demand	6	10	15	4	

Soln

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	14	5	14
O_2	6	9	1	7	16
O_3	4	1	6	4	5
Demand	6	10	15	4	35

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(Q)

	D_1	D_2	D_3	D_4	Supply
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
Demand	5	8	7	14	34

Min c_{ij} = 8

	D_1	D_2	D_3	D_4	Supply
O_1	19 x	x 30	50 x	10 $\boxed{7}$	$\cancel{7}$ / 0
O_2	70 $\boxed{2}$	x 30	40 $\boxed{7}$	60 x	$\cancel{9}$ / $\cancel{2}$ / 0
O_3	40 $\boxed{3}$	8 $\boxed{8}$	70 x	20 $\boxed{7}$	$\cancel{18}$ / $\cancel{10}$ / 3
Demand	$\cancel{5}$ / 3	$\cancel{8}$ / 0	$\cancel{7}$ / 0	$\cancel{14}$ / $\cancel{7}$ / 0	

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Vogel's Approximation Method (VAM)

Algorithm (Penalty Method).

(1) Check the T.P is balanced or not if not Convert into balanced form.

(ii) Findout the penalty cost for each row and Column. (penalty cost means the difference between min unit cost and the next min. unit cost)

(iii) Choose the maximum penalty cost from row and Column. If it is more than one, Choose any one arbitrarily.

(iv) Give the allocation to the shell which is having min. unit Cost from the row or the Column, that is responsible for Maximum penalty value.

Allocate the min. value among the Supply and the demand to that shell.

(5) Delete the row or column which satisfy the unit of Supply or Demand.

(6) Continue this process until it satisfies the row and Column requirement i.e Supply and Demand =0

(7) Write the Model Summary.

(a)

	D_1	D_2	D_3	D_4	Supply	penalty
O_1	19	30 X	50	10	7	20 (30-10)
O_2	70	30 X	40	60	9	10
O_3	40	8 $\boxed{8}$	70	20	~~18~~/10	12
Demand	5	8/0	7	14		
penalty	21	22 ↑	10	10		

					Penalty				
19 $\boxed{5}$	50 X	10 $\boxed{2}$	7/2/0	9	40	40	—	—
70 X	40 $\boxed{7}$	60 $\boxed{2}$	9/7/0	20	20	20	20	40
40 X	70 X	20 $\boxed{10}$	10/0	20	50 ←	—	—	—
5/0	7/0	14/4/2/0						

penalty		
21 ↑	10	10
—	10	10
—	10	50 ↑
	40	60 ↑
	40	—

Min Transportation Cost is =

$(8\times8) + (19\times5) + (10\times2) + (40\times7) + (60\times2) + (20\times10) = 779$

Optimality Test :-

Non-Degenerate Solⁿ - If the no. of allocation in the Transportation problem satisfies $N = m+n-1$, then the solution is Non-degenerate solution.

Where, N = No. of allocation
m = No. of source
n = No. of destination

Degenerate Solution - If the no. of allocation (N) in the TP satisfies $N < m+n-1$, then the solution is degenerated Solⁿ.

→ This solⁿ can not give the optimal value.

Modified Distribution Method (MODI)

Step-1: Find initial basic feasible solution of TP by any of the three Method.

Step-2: Findout u_i and v_j for row and column which satisfy $u_i + v_j = c_{ij}$ for each occupied cell. To start this activity we assign '0' to any row and column having maximum number of allocation. If max^m allocation is more than one choose any one.

Step-3: For each unoccupied cell, find $u_i + v_j$ and write at the bottom left corner of that cell.

Step-4: Find Δ_{ij} for each unoccupied cell by applying formula $\Delta_{ij} = c_{ij} - (u_i + v_j)$ and write the value at bottom right corner of that cell. and check optimality.

(i) If all $\Delta_{ij} > 0$, then the solution is optimal and unique solution exists.

(ii) If all $\Delta_{ij} \geq 0$, then the solution is optimum, but an alternate solution exists.

(iii) If at least one $\Delta_{ij} < 0$, the solution is not optimum. In this case go to the next step.

Step-5 : Select the empty cell having the most negative value of Δ_{ij}. From that cell draw a closed path by drawing horizontal and vertical lines with the corner cell occupied. Assign +ve and -ve sign alternately and find the minimum allocation from the cell having -ve sign. This allocation should be added to the allocation having +ve sign and substracted from the allocation having -ve sign.

Step-6 : The above step gives a better solution by making one or more occupied cell as empty and one empty cell as occupied. For this new set of basic feasible allocation repeat from step-2 onwards till an optimum basic feasible solution is obtained

(Q) Obtain the initially feasible solution to the following transportation problem. Is that solution is optimal solution? If not then obtain the optimal solⁿ.

			Supply
2	7	4	5
3	3	1	8
5	4	7	7
1	6	2	14
Demand 7	9	18	

Solⁿ

			Supply	penalty				
2 [5]	7 ×	4 ×	~~5~~/0	2 ←	–	–	–	–
3 ×	3 ×	1 [8]	~~8~~/0	2	2 ←	–	–	–
5 ×	4 [7]	7 ×	~~7~~/0	1	1	1	4	4
1 [2]	6 [2]	2 [10]	14/~~4~~/2/0	1	1	1	5 ←	6 ←

Demand: ~~7~~/2/0, ~~9~~/7/0, ~~18~~/10/0

penalty:

1	1	1
2	1	1
4	2	5 ↑
~~50~~	~~40~~	~~20~~ ↑
↑ 4	2	–
–	2	–
	↑ 4	

Total Transportation Cost =

$(2\times5) + (1\times8) + (4\times7) + (1\times2) + (6\times2) + (10\times2)$

$= 80$

~~Step-2~~ As the no. of non-negative allocation = $m+n-1$

$$\Rightarrow 6 = 4+3-1 = 6$$

So it is eligible for optimality test.

Step-2 : Net Evaluation for assigned cell ($u_i + v_j = c_{ij}$)

	$V_1=1$	$V_2=6$	$V_3=2$
$U_1=1$	2		
$U_2=-1$			1
$U_3=-2$		4	
$0=U_4$	1	6	2

Let $u_4 = 0$ bcoz u_4 row have maximum no. of allocation.

$u_1 + v_1 = 2 \Rightarrow u_1 + 1 = 2 \Rightarrow u_1 = 1$

$u_2 + v_3 = 1 \Rightarrow u_2 + 2 = 1 \Rightarrow u_2 = -1$

$u_3 + v_2 = 4 \Rightarrow u_3 + 6 = 4 \Rightarrow u_3 = -2$

$u=0$ $\quad u_4 + v_1 = 1 \Rightarrow 0 + v_1 = 1 \Rightarrow v_1 = 1$

$u_4 + v_2 = 6 \Rightarrow 0 + v_2 = 6 \Rightarrow v_2 = 6$

$u_4 + v_3 = 2 \Rightarrow 0 + v_3 = 2 \Rightarrow v_3 = 2$

Step-3 Net Evaluation for unassigned cell / Empty cell.

	$V_1=1$	$V_2=6$	$V_3=2$
$U_1=1$		7	4
$U_2=-1$	3	3	
$U_3=-2$	5		7
$U_4=0$			

$\Delta_{ij} = C_{ij} - (U_i + V_j)$

$\Delta_{12} = 7 - (1+6) = 0$

$\Delta_{13} = 4 - (1+2) = 1$

$\Delta_{21} = 3 - (-1+1) = 3$

$\Delta_{22} = 3 - (-1+6) = \underline{\underline{-2}}$

~~$\Delta_{331} =$~~

$\Delta_{31} = 5 - (-2+1) = 6$

$\Delta_{33} = 7 - (-2+2) = 7$

As $(\Delta_{22} < 0)$ so it is not an optimal solution.

Step-4

(Take the quantity)

5		
	$+\theta$	$8-\theta$
	7	
2	$2-\theta$	$10+\theta$

(clock wise directions)

* Δ_{22} having the negative value so first select the cell and ~~assign + sign~~, draw a closed path (The corner cell must be occupied cell)

→ Find Minimum allocation from the cell having -ve sign.

$\text{Min}(8-\theta,\ 2-\theta) = 0$

$\cancel{\theta = 8}$ & $\boxed{\theta = 2}$ → Min

Take Minimum value of θ.

Step-5 A New assigned table

5		
	2	6
	7	
2		12

No. of assigned cell = 6

$M + n - 1 = 6$

$\Rightarrow 4 + 3 - 1 = 6$

So it is eligible for optimality.

Step-6 Net Evaluation for assigned cell ($U_i + V_j = C_{ij}$)

	$V_1 = 0$	$V_2 = 3$	$V_3 = 1$
$U_1 = 2$	2		
$U_2 = 0$		3	1
$U_3 = 1$		4	
$U_4 = 1$	1		2

Let $U_2 = 0$

$U_1 + V_1 = 2 \Rightarrow U_1 + 0 = 2 \Rightarrow U_1 = 2$

$U_2 + V_2 = 3 \Rightarrow 0 + V_2 = 3 \Rightarrow V_2 = 3$

$U_2 + V_3 = 1 \Rightarrow 0 + V_3 = 1 \Rightarrow V_3 = 1$

$U_3 + V_2 = 4 \Rightarrow U_3 + 3 = 4 \Rightarrow U_3 = 1$

$U_4 + V_1 = 1 \Rightarrow 1 + V_1 = 1 \Rightarrow V_1 = 0$

$U_4 + V_3 = 2 \Rightarrow U_4 + 1 = 2 \Rightarrow U_4 = 1$

Step-7 Net Evaluation for unassigned cell.

	$V_1=0$	$V_2=3$	$V_3=1$
$u_1=2$		7	4
$u_2=0$	3		
$u_3=1$	5		7
$u_4=1$		6	

$\Delta_{ij} = C_{ij} - (u_i+v_j)$

$\Delta_{12} = 7-(2+3) = 2$

$\Delta_{21} = 3-(0+0) = 3$

$\Delta_{31} = 5-(1+0) = 4$

$\Delta_{33} = 7-(1+1) = 5$

$\Delta_{42} = 6-(1+3) = 2$

As all $(\Delta_{ij} \geq 0)$ so it is an optimal Solution.

2 [5]	7	4
3	3 [2]	1
5	4 [7]	7
1 [2]	6	12 [2]

Transportation Cost is =

$(2\times5)+(3\times2)+(4\times7)+(1\times2)+(12\times2) = 76/-$

~~Types of Solution~~ in

Degeneracy in Transportation Problem -

→ In T.P, if the no. of non-negative independent allocation is less than $m+n-1$, there exists a degeneracy. ✓

→ To resolve the degeneracy follow following steps -

(i) Among the empty cell we choose the empty cell having the least cost which is of an independent position.

(ii) To the cell as choosen allocate a small positive quantity.

(Q) Obtain the initial basic feasible solution by using North-west Corner Rule and check the optimality test and check optimality solution. If not then find the optimal solution.

Ans- (Q)

	A	B	C	Supply
	2	2	3	10
	4	1	2	15
	1	3	1	40
Demand	20	15	30	

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Soln

Step-1

	A	B	C	Supply
1	2 [10]	2 ×	3 ×	10/0
2	4 [10]	1 [5]	2 ×	15/5/0
3	1 ×	3 [10]	1 [30]	40/30
Demand	20/10/0	15/10/0	30	

Total transportation cost = $(2\times10)+(4\times10)+(1\times5)+(3\times10)+(1\times30)$

$= 125/-$

Step-2 No. of assigned cell = 5

$M+n-1 = 3+3-1 = 5$

Net evaluation for assigned cell

$C_{ij} = u_i + v_j$

	$v_1=4$	$v_2=1$	$v_3=-1$
$u_1=-2$	2		
$u_2=0$	4	1	
$u_3=2$		3	1

$u_2 = 0$

$u_1+v_1=2 \Rightarrow u_1=-2$

$u_2+v_1=4 \Rightarrow v_1=4$

$u_2+v_2=1 \Rightarrow v_2=1$

$u_3+v_2=3 \Rightarrow u_3=2$

$u_3+v_3=1 \Rightarrow v_3=-1$

put $u_2=0$

Step 3 Net Evaluation for non-assigned cell.

$\Delta_{ij} = C_{ij} - (u_i+v_j)$

	$v_1=4$	$v_2=1$	$v_3=-1$
$u_1=-2$		2	3
$u_2=0$			2
$u_3=2$	1		

$\Delta_{12} = 2-(-2+1) = 3$

$\Delta_{13} = 3-(-2-1) = 6$

$\Delta_{23} = 2-(0-1) = 3$

$\Delta_{31} = 1-(2+4) = -5$

$\Delta_{ij} \leq 0$

$\Delta_{31} < 0$ So it is not an optimal soln

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Step-4

10		
$-\theta$ 10	$+\theta$ 5	
$+\theta$	$10-\theta$	30

* Δ_{31} having Negative value so allocat C_{31} first.

$$\text{Min}\,(10-\theta,\,10-\theta)=0$$

$$10-\theta=0$$

$$\boxed{\theta=10}$$

Step-5 A New assigned table -

10		
	15	
10		30

Now check optimality test.

$M = M+n-1$

$\Rightarrow\ 4 = 3+3-1=5$

As the no. of assigned cell $\neq (m+n-1)$ so it is a degeneracy problem.

Step-6 Among the empty cell we choose the empty cell having least cost.

2 [10]	2 [ε]	3
4	1 [15]	2
1 [10]	3	1 [30]

(least value)

(Allocate a small +ve quantity)

No. of assigned cell = 5

$m+n-1 = 3+3-1 = 5$

So it is eligible for optimality test.

for assigned cell

	$V_1=2$	$V_2=2$	$V_3=2$
$u_1=0$	2	2 [ε]	
$u_2=-1$		1	
$u_3=-1$	1		1

$u_1+v_1 = 2 \Rightarrow V_1 = 2$

$u_1+V_2 = 2 \Rightarrow V_2 = 2$

$u_2+V_2 = 1 \Rightarrow u_2 = -1$

$u_3+V_1 = 1 \Rightarrow u_3 = -1$

$u_3+V_3 = 1 \Rightarrow V_3 = 2$

Put $\boxed{u_1 = 0}$.

For unassigned cell

	$V_1=2$	$V_2=2$	$V_3=2$
$u_1=0$			3
$u_2=-1$	4		2
$u_3=-1$		3	

$\Delta_{ij} = C_{ij} - (u_i + v_j)$

$\Delta_{13} = 3-(0+2) = 1$

$\Delta_{21} = 4-(-1+2) = 3$

$\Delta_{23} = 2-(-1+2) = 1$

$\Delta_{32} = 3-(-1+2) = 2$

As the $\Delta_{ij} \geq 0$, so it is optimal soln.

2 [10]	2 [ε]	3
4	1 [15]	2
1 [10]	3	1 [30]

Total transportation Cost $= (2\times10) + (1\times15) + (2\times\varepsilon) + (1\times10) + (1\times30)$

$= 75/-$

Maximization Case in transportation Problem:-

Objective - Maximise the total profit.

→ We have to convert the maximisation problem into minimisation by substracting all the elements in the given transportation table from the highest element.

(Q) Solve the following transportation problem to maximise the profit.

Destination

Source	A	B	C	D	Supply
1	15	51	42	33	23
2	80	42	26	81	44
3	90	40	66	60	33
Demand	23	31	16	30	100

Higher value

Soln The given problem is to maximize the profit, we convert it into minimse.

→ For minimisation type, we substract all the elements from the highest element 90.

	A	B	C	D	Supply
1	75	39	48	57	23
2	10	48	64	9	44
3	0	50	24	30	33
Dem	23	31	16	30	

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						Penalty				
1	75 X	39 [23]	48 X	57 X	23/0	9	18	18	18	39
2	10 [6]	48 [8]	64 X	9 [30]	44/38/8/0	1	1	1	39	48 ←
3	0 [17]	50 X	24 [16]	30 X	33/17/0	6	30 ←	–	–	–
Demand	23/6/0	31/23/0	16/0	30/0						

penalty

10	9	24 ↑	21
10	9	–	21
↑ 65	11	–	48
–	11	–	48 ↑
–	11	–	–

The initial basic feasible solution =

No. of allocated Cell = M+N-1

$\Rightarrow 6 = 3+4-1 = 6$

The Solution is Non-degenerate Soln.

<u>Optimality test Using MODI Method</u> -

Net Evaluation for allocated Cell -

	$V_1 = 10$	$V_2 = 48$	$V_3 = 34$	$V_4 = 9$
$u_1 = -9$		39 [23]		
$u_2 = 0$	10	48		9
$u_3 = -10$	0		24	

$\boxed{u_2 = 0}$

$C_{ij} = u_i + v_j$

$u_1 + v_2 = 39 \Rightarrow u_1 = -9$

$u_2 + v_1 = 10 \Rightarrow v_1 = 10$

$u_2 + v_2 = 48 \Rightarrow v_2 = 48$

$u_2 + v_4 = 9 \Rightarrow v_4 = 9$

$u_3 + v_3 = 24 \Rightarrow v_3 = 34$

$u_3 + v_1 = 0 \Rightarrow u_3 = -10$

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Net Evaluation for non-allocated cell -

	$V_1=10$	$V_2=48$	$V_3=34$	$V_4=9$
$U_1=-9$	75		48	57
$U_2=0$			64	
$U_3=-10$		50		30

$\Delta ij = C_{ij} - (U_i + V_j)$

$\Delta_{12} = 75 - (-9+10) = 74$

$\Delta_{13} = 48 - (-9+34) = 33$

$\Delta_{14} = 57 - (-9+9) = 57$

$\Delta_{23} = 64 - (0+34) = 30$

$\Delta_{32} = 50 - (-10+48) = 12$

$\Delta_{34} = 30 - (-10+9) = 31$

All $\Delta_{ij} \geq 0$

The optimum profit $= 23 \times 51 + 6 \times 80 + 8 \times 42 + 81 \times 30 + 90 \times 17 + 25 \times 16 +$

$= 7005/-$

Stepping - Stone Method -

→ This method is an optimization technique used to find optimal transportation cost.
→ In this method we form closed path for every unoccupied cell.

1. Check for degeneracy by formula $(m+n-1)$
2. forming loops / paths for unoccupied cell in such a way that at each corner point of loop, there must be an allocated cell.
3. Assign +ve, -ve alternately and add +1 where +ve symble and substract -1 where -ve symbol.
4. find out increase in transportation cost per unit,
5. If all cost value ≥ 0, the solution is optimal. If any total cost value is -ve then we need re-allocation and proceed for next step.
6. find the most negative cost and find the loop & draw a separate table. If we have two or more same most negative cost value then draw two separate loop and evaluate them separately.
7. for allocation find the smallest allocation in the path of loop, by considering only corner cell. Add the value with +ve cell & substract from -ve cell.

Q) find the initial basic solution whether they are optimal or not by using stepping stone method if not findout the optimal solution.

					Supply
	1	2	1	4	30
	2	3	2	1	50
	4	2	5	9	20
Demand	20	40	30	10	

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					Penalty
1 [20]	2 [10]	1 ×	4 ×	30/10/0	1 1 1 1 —
2 ×	3 [10]	2 [30]	1 [10]	50/40/30/0	1 1 1 1 1 2
4 ×	2 [20]	5 ×	9 ×	20/0	2 2 — — —
20/0	40/20/10/0	30/0	10/0		
Penalty 1	1	1	3↑		
1	1	1	—		
↑1	1	1	—		
—	↑1	1	—		
—	3↑	2	—		
	—	2			

Total Transportation cost $= (1\times20) + (2\times10) + (3\times10) + (2\times30) + (1\times10) + (2\times20)$

$= 180/-$

No. of assigned cell $= 6$

$m+n-1 = 6$

find all the index

Step-2

1 [20]	2 [10] −	1 +	4
2	3 [10] +	2 [30] −	1 [10]
4	2 [20]	5	9

$I_{13} = +1 - 2 + 3 - 2 = 0$

1 [20]	2 [10] −	1	4 +
2	3 [10] +	2 [30]	1 [10] −
4	2 [20]	5	9

$I_{14} = 4 - 1 + 3 - 2 = 4$

1 [20]	2 [10]	1	4
2	3 [10]	2 [30]	1 [40]
4	2 [20]	5	9

$$I_{21} = +2 - 1 + 2 - 3 = 0$$

1 [20]	2 [10]	1	4
2	3 [10]	2 [30]	1 [40]
4	2 [20]	5	9

$$I_{31} = 4 - 1 + 2 - 2 = 3$$

1 [20]	2 [10]	1	4
2	3 [10]	2 [30]	1 [40]
4	2 [20]	5	9

$$I_{33} = 5 - 2 + 3 - 2 = 4$$

1 [20]	2 [10]	1	4
2	3 [10]	2 [30]	1 [40]
4	2 [20]	5	9

$$I_{34} = 9 - 2 + 3 - 1 = 9$$

All the index value +ve then 180 is optimal solution.

Topic:

Assignment Problem

Assignment Problem

Suppose there are 'n' jobs and 'n'-persons are available for doing the job. Assume that each person can do a single job at a time. Let

C_{ij} - Cost of the ith person is assigned to the jth job.

The problem is to find an assignment (which job should be assigned to which person, on one-to-one basis), so that the total cost for performing the job is minimum.

It can be stated in $n \times n$ cost matrix $[C_{ij}]$ of real numbers as given in the following table -

$$
\begin{array}{cc}
 & \text{Jobs} \\
 & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & j \cdots & n \end{array} \\
\text{persons} \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} &
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & \cdots & C_{1j} & \cdots & n \\
C_{21} & C_{22} & C_{23} & \cdots & C_{2j} & \cdots & C_{2n} \\
C_{31} & C_{32} & C_{33} & \cdots & C_{3j} & \cdots & C_{3n} \\
\vdots & \vdots & & & & & \\
C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nj} & \cdots & C_{nn}
\end{bmatrix}
\end{array}
$$

Mathematical formulation -

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_{ij}$$

subjected to,

$$x_{ij} = \begin{cases} 1, & \text{if } i\text{th person is assigned } j\text{th job} \\ 0, & \text{if not} \end{cases}$$

and $\sum_{i=1}^{n} x_{ij} = 1$ (One job is done by ith person)

and $\sum_{j=1}^{n} x_{ij} = 1$ (only one person is assigned with jth job)

and x_{ij} denotes jth job assigned with ith person.

Hungarian Method for assignment problem:-

Step-1 : Balance checking (No. of rows = No. of Column)

Step 2 : Standard (Minimisation)
e.g.- Cost, westage etc

→ If the problem is given in terms of maximization (profit, production quantity etc).

* Convert the maximization problem to minimization (Identify the maximum cell value and substract all the cell values from it so that it will convert into minimization form.

Step-3 : Opportunity cost table.
(i) Row operation
(ii) Column operation

Row operation -
Locate the smallest element in each row of the given assignment problem and substrac from each element of that corresponding Row i.e called Row operation.

Column operation -
Locate the smallest element in each column and then substract from each element of the corresponding column so that it is called Column operation.

Now every row and every column have at least one zero.

Step-4 Assignment
(i) Examine the rows successfully untill a row with exactly one unmarked zero is obtained make a assignment to the

single '0' by making square (□) around it.

(2) For each ~~row~~ zero (0) value which has assigned make it cross-out (x) to all other zeros in the same rows and column.

(3) Repeat point (1) & (2) for each column and row also with exactly single zero value cell.

(4) If a row or column has two unknown unmark zero then choose the zeros arbitarily

(5) Continue this process until ~~there~~ all the zeros in the row or column are either enclosed by (□) square or (x) symbol.

Step 5 Optimality test -
If the no. of assigned cell is equal to the no. of rows or column then it is an optimal solution. If not then go to the step-6.

Step-6
Draw a ~~set~~ minimum no. of horizontal and vertical lines to cover all the zeros in the revised table obtain from the step-5 by using the following procedure.

(1) Tick (✓) the rows they don't have any mark (□) zeros.

(2) Examine the mark rows (✓). If any zero occurs in any row make a (✓) to the corresponding column that contains those assigned zeros.

(3) Examine the mark column. If any assigned ~~column~~ Zero (0) occurs in those column then (✓) the respective rows that contains assigned zeros.

(4) Repeat this process untill No more rows or column can be marked.

(5) Draw the setup horizontal line through each unmarked row and marked column (URMC).

Step-7

Develop the new revised opportunity cost table among the cell not covered by any line (Choose the smallest element let say the value is 'k' i.e called key element.

* (1) Substract 'k' from every element in the cell not covered by any line.

* (2) Add 'k' to every element in the cell covered by two lines i.e intersection of two lines.

* (3) Element in the cell covered by one line will remain unchanged.

Repeat step 3 to step 7 untill an optimal solution is obtained.

+ → add

− → same

▭ uncovered → Substract

(Q) A Company has 5 Machines on which to do 5 Jobs. Each job can be assigned to one and only one machine. The following cost values for each job to each machine is given in the following table. What are the job assigned which will minimise the cost.

Jobs	A	B	C	D	E
	Machine				
1	13	8	16	18	19
2	9	15	24	9	12
3	12	9	4	4	4
4	6	12	10	8	13
5	15	17	18	12	20

Sol^n Step 1 Balanced – No. of Rows = 5, No. of Column = 5

Step-2 : Minimization form

Step-3 Select the smallest element in each row and subtract this smallest element from all the elements in its row.

Row operation

	A	B	C	D	E
1	5	0	8	10	11
2	0	6	15	0	3
3	8	5	0	0	0
4	0	6	4	2	7
5	3	5	6	0	8

Column operation

	A	B	C	D	E
1 M_1	5	0	8	10	11
2 M_2	0	6	15	0	3
3 M_3	8	5	0	0	0
4 M_4	0	6	4	2	7
5 M_5	3	5	6	0	8

Step 4

	A	B	C	D	E
1	5	$\boxed{0}$	8	10	11
2	~~0~~	6	15	~~0~~	3
3	8	5	$\boxed{0}$	~~0~~	~~0~~
4	$\boxed{0}$	6	4	2	7
5	3	5	6	$\boxed{0}$	8

No. of assigned zero = 4

Step 5

No. of rows / Column = 5

4 ≠ 5

As 4≠5 so it is not a optimal Solution

Step-6

5	$\boxed{0}$	8	10	11
~~0~~	6	15	~~0~~	3
8	5	$\boxed{0}$	~~0~~	~~0~~
$\boxed{0}$	6	4	2	7
3	5	6	$\boxed{0}$	8

(i) (i) (i) (ii) (iii)

(RMC)

Step-7 Develop the new revised opportunity Cost table.

choose Minimum Cost among the cell not Covered. Here 3 is Smallest Cost.

	A	B	C	D	E
1	8	$\boxed{0}$	8	13	11
2	~~0~~	3	12	~~0~~	$\boxed{0}$
3	11	5	$\boxed{0}$	3	~~0~~
4	$\boxed{0}$	3	1	2	4
5	3	2	3	$\boxed{0}$	5

No. of assigned zero = ~~4~~ 5

No. of rows / Column = 5

No. of assigned zero = No. of rows/Column

Hence solution is optimal.

Optimal assignment and optimum cost of assignment.

Job	Machine	Cost
1	B	8
2	E	12
3	C	4
4	A	6
5	D	12
		42/-

Ex Using the following cost Matrix, determine (a) optimal Job assignment
(b) The cost of assignments

Job

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \text{Mechanic} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 10 & 3 & 3 & 2 & 8 \\ 9 & 7 & 8 & 2 & 7 \\ 7 & 5 & 6 & 2 & 4 \\ 3 & 5 & 8 & 2 & 4 \\ 9 & 10 & 9 & 6 & 10 \end{bmatrix} \end{array}$$

Step-1 Balanced

2019 Step 2 Select the smallest ~~value~~ element in each row and substract this smallest element from all the elements in its row.

Row operation

$$\begin{bmatrix} 8 & 1 & 1 & 0 & 6 \\ 7 & 5 & 6 & 0 & 5 \\ 5 & 3 & 4 & 0 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 4 & 3 & 0 & 4 \end{bmatrix}$$

Column operation

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 7 & 0 & 0 & 0 & 4 \\ 6 & 4 & 5 & 0 & 3 \\ 4 & 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 2 & 3 & 2 & 0 & 2 \end{bmatrix} \end{array}$$

Modified Matrix is -
$$\begin{bmatrix} 7 & 0 & 0 & 0 & 4 \\ 6 & 4 & 5 & 0 & 3 \\ 4 & 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 2 & 3 & 2 & 0 & 2 \end{bmatrix}$$

Step-3

In this modified matrix we draw the minimum no. of lines to cover all zeros (horizontal or vertical)

$$\begin{bmatrix} 7 & 0 & 0 & 0 & 4 \\ 6 & 4 & 5 & 0 & 3 \\ 4 & 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 2 & 3 & 2 & 0 & 2 \end{bmatrix}$$

Number of lines drawn to cover all zeros is $4 = N$

The order of matrix is $n = 5$

Hence $N < 5$

Now we get the second modified matrix by substracting the smallest uncovered element from the remaining uncovered elements and adding it to the elements intersection of lines.

Assignment

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}\begin{bmatrix} 9 & \boxed{0} & \cancel{0} & 2 & 6 \\ 6 & 2 & 3 & \boxed{0} & 3 \\ 4 & \cancel{0} & 1 & \cancel{0} & \boxed{0} \\ \boxed{0} & \cancel{0} & 3 & \cancel{0} & \cancel{0} \\ 2 & 1 & \boxed{0} & \cancel{0} & 2 \end{bmatrix}$$

Choose the line drawn to cover all zeros.

No. of lines drawn to cover all zeros $N=5$

The order of matrix is $n=5$

Hence $N = n$.

Assignment

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 9 & \boxed{0} & \cancel{0} & 2 & 6 \\ 6 & 2 & 3 & \boxed{0} & 3 \\ 4 & \cancel{0} & 1 & \cancel{0} & \boxed{0} \\ \boxed{0} & \cancel{0} & 3 & \cancel{0} & \cancel{0} \\ 2 & 1 & \boxed{0} & \cancel{0} & 2 \end{bmatrix} \end{array}$$

MECHANIC

optimal assignment and optimum Cost of assignment -

Job	Mechanic	Cost
1	D	3
2	A	3
3	E	9
4	B	2
5	C	4
		21/-

(Q) four different Jobs can be done on four different machines and the take down time costs are prohibitively high for change overs. The matrix below gives the cost in rupees for producing Job i on the machine j.

Jobs	Machines			
	M_1	M_2	M_3	M_4
J_1	5	7	11	6
J_2	8	5	9	6
J_3	4	7	10	7
J_4	10	4	8	3

How should the Jobs be assigned to the various machines so that the total cost is minimized?

Sol^n Substract the least element in respective row.

Row oper^n

$$\begin{array}{c|c} & \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{array}{c} J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} & \begin{bmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{bmatrix} \end{array}$$

Column oper^n

$$\begin{array}{c|c} & \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{array}{c} J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} & \begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Now we draw minimum no. of lines to Cover all zeros.

$$\begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix}$$

No. of lines drawn to Cover all zeros = 3

order of Matrix = 4

2nd modified Matrix,

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & 2 \\ 0 & 2 & 1 & 3 \\ 7 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 8 & 0 & 0 & 0 \end{bmatrix}$$

$N = 3 < n = 4$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 8 & 0 & 0 & 0 \end{bmatrix}$$

$N = 4 = n = 4$

Hence we can make an assignment.

$$\begin{array}{c|cccc} & M_1 & M_2 & M_3 & M_4 \\ \hline J_1 & \cancel{0} & \cancel{0} & \cancel{0} & \boxed{0} \\ J_2 & 5 & \boxed{0} & \cancel{0} & 2 \\ J_3 & \boxed{0} & 1 & \cancel{0} & 2 \\ J_4 & 8 & \cancel{0} & \boxed{0} & \cancel{0} \end{array}$$

Since no rows and no columns have single zero, we have a different assignment (Multiple Solution)

optimal assignment

Job	Machine
J_1	M_4
J_2	M_2
J_3	M_1
J_4	M_3

Minimum (Total Cost) $\Rightarrow 6 + 5 + 4 + 8 = 23/-$

Alternate Solution

$J_1 \to M_1$, $J_2 \to M_2$, $J_3 \to M_3$, $J_4 \to M_4$

Minimum $\Rightarrow 5 + 5 + 10 + 3 = 23/-$

Unbalanced Assignment problem:-

Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix.

→ To make it balanced, we add a dummy row or column with all the entries as zero.

ex There are four Jobs to be assigned to five machines. The amount of time in hours required for the jobs per machine are given in the following matrix.

Jobs	A	B	C	D	E
1	4	3	6	2	7
2	10	12	11	14	16
3	4	3	2	1	5
4	8	7	6	9	6

(Machines: A–E)

Find an optimum assignment of jobs to the machines to minimize the total processing time and also find out for which machine no jobs is assigned. What is the total processing time to complete all the jobs?

soln Since the cost matrix is not a square matrix, the problem is unbalanced. We added a dummy job with corresponding entries '0'.

$$\begin{bmatrix} 4 & 3 & 6 & 2 & 7 \\ 10 & 12 & 11 & 14 & 16 \\ 4 & 3 & 2 & 1 & 5 \\ 8 & 7 & 6 & 9 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row operation

$$\begin{bmatrix} 2 & 1 & 4 & 0 & 5 \\ 0 & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-4

$$\begin{bmatrix} 2 & 1 & 4 & \boxed{0} & 5 \\ \boxed{0} & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & \cancel{0} & 4 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Assigned (N) = 3

Row operⁿ

$$\begin{bmatrix} 2 & 1 & 4 & 0 & 5 \\ 0 & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Assignment

Step-4

$$\begin{bmatrix} 2 & 1 & 4 & \boxed{0} & 5 \\ \boxed{0} & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & \cancel{0} & 4 \\ 2 & 1 & \boxed{0} & 9 & \cancel{0} \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

$N = 4$

$n = 5$

$N \neq n$

As $N \neq n$ so it is not a optimal solution

Step-5

$$\begin{bmatrix} 2 & 1 & 4 & \boxed{0} & 5 \\ \boxed{0} & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & \cancel{0} & 4 \\ 2 & 1 & \boxed{0} & 9 & \cancel{0} \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

(URMC)

Min uncovered = 1

Step-6

1 0 3 0 4

0 2 1 5 5

Step-6

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1 & \boxed{0} & 3 & \cancel{0} & 4 \\ \boxed{0} & 2 & 1 & 5 & 6 \\ 2 & 1 & \cancel{0} & \boxed{0} & 3 \\ 2 & 1 & \boxed{0} & 10 & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & 1 & \boxed{0} \end{bmatrix} \end{array}$$

Job	Machines	Cost
1	B	3
2	A	10
3	D	1
4	C	6
5	E	0
		Rs. 20/-

Maximization in Assignment Problem :-

Objective — To maximize the Profit

Step - Convert the given matrix into the loss matrix by substracting all the elements from the highest element.

We apply the steps in Hungarian method to get the optimum assignment.

(Q) A marketing manager has 5 salesmen and there are 5 sales districts. Consider the capabilities of the sales man and the nature of districts, the estimates made by the marketing manager for the sales per month (in 1,000 rupees) for each salesman in each district would be as follows.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
32 & 38 & 40 & 28 & 40 \\
40 & 24 & 28 & 21 & 36 \\
41 & 27 & 33 & 30 & 37 \\
22 & 38 & 41 & 36 & 36 \\
29 & 33 & 40 & 35 & 39
\end{bmatrix}
\end{array}
$$

Find the assignment of salesmen to the districts that will result in the maximum sale.

Soln

$$
\begin{array}{c c}
 & \begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
9 & 3 & 1 & 13 & 1 \\
1 & 17 & 13 & 20 & 5 \\
0 & 14 & 8 & 11 & 4 \\
19 & 3 & 0 & 5 & 5 \\
12 & 8 & 1 & 6 & 2
\end{bmatrix}
\end{array}
$$

Row operⁿ

$$
\begin{bmatrix}
8 & 2 & 0 & 12 & 0 \\
0 & 16 & 12 & 19 & 4 \\
0 & 14 & 8 & 11 & 4 \\
19 & 3 & 0 & 5 & 5 \\
11 & 7 & 0 & 5 & 1
\end{bmatrix}
$$

Column operⁿ

$$
\begin{bmatrix}
8 & 0 & 0 & 7 & 0 \\
0 & 14 & 12 & 14 & 4 \\
0 & 12 & 8 & 6 & 4 \\
19 & 2 & 0 & 0 & 5 \\
11 & 5 & 0 & 0 & 1
\end{bmatrix}
$$

Assignment

$$
\begin{bmatrix}
8 & \boxed{0} & \cancel{0} & \cancel{7} & \cancel{0} \\
\boxed{0} & 14 & 12 & 14 & 4 \\
\cancel{0} & 12 & 8 & 6 & 4 \\
19 & 1 & \boxed{0} & \cancel{0} & 5 \\
11 & 5 & \cancel{0} & \boxed{0} & 1
\end{bmatrix}
$$

$N \neq n$

$4 \quad 5$

Step – 5

$$\begin{bmatrix} 8 & 0 & 0 & 6 & 0 \\ 0 & 4 & 12 & 14 & 4 \\ 0 & 12 & 8 & 6 & 4 \\ 19 & 1 & 0 & 0 & 5 \\ 11 & 5 & 0 & 0 & 1 \end{bmatrix}$$

Step-5

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 12 & \boxed{0} & \cancel{0} & 7 & \cancel{0} \\ \boxed{0} & 10 & 8 & 10 & \cancel{0} \\ \cancel{0} & 8 & 4 & 2 & \boxed{0} \\ 19 & 1 & \boxed{0} & \cancel{0} & 5 \\ 11 & 5 & \cancel{0} & \boxed{0} & 1 \end{bmatrix} \end{array}$$

Salesman	Sales District	Cost
1	B	38
2	A	40
3	E	37
4	C	41
5	D	35
		Rs 191/-

Topic:

Integer Programming(Branch And Bound Method)

Integer Programming :-

A linear programming problem in which all or some of the values of decission Variable are assumed to be non-negative integer then that is called integer programming problem.

(i) pure IPP

(ii) Mixed-type IPP

→ Used to Convert fractional Value into an integer.

→ It can be solved by 2 method -

(i) Cutting plane algorithm

(ii) Branch and Bound algorithm.

Branch and Bound algorithm :-

Step-1 : First solve the problem by ignoring the integrality condition.

Step-2 : If the solution is an integer then the Current solution is optimum for the given integer programming problem.

(1) If the solution is not integer

Let x_r is not an integer value.

$$x_r^* \le x_r \le x_{r+1}^*$$

$$x_r \le x_r^* \quad , \quad x_r \ge x_{r+1}^*$$

Let $\overset{x}{1 \le 1.5 \le 2}$ → Bound.

→ upper bound

Lower bound $1.5 \le 1 \quad , \quad 1.5 \ge 2$

$$x \le 1 \quad , \quad x \ge 2$$

→ Can solve by graphical or Simplex.

(Q) Use branch and bound technique to solve the following:

$$\text{Max } Z = x_1 + 4x_2$$

s.t.

$$2x_1 + 4x_2 \le 7$$

$$5x_1 + 3x_2 \le 15$$

$x_1, x_2 \ge 0$ and are integers.

Soln Solve the LPP by graphical method -

$$\text{Max } Z = x_1 + 4x_2$$

$$2x_1 + 4x_2 = 7 \quad \text{--- (i)}$$

$$5x_1 + 3x_2 = 15 \quad \text{--- (ii)}$$

$$x_1, x_2 \ge 0$$

from eqn (i) -

$$2x_1 + 4x_2 = 7$$

$x_1 = 0, \; x_2 = 7/4 = 1.75$

$x_2 = 0, \; x_1 = 7/2 = 3.5$

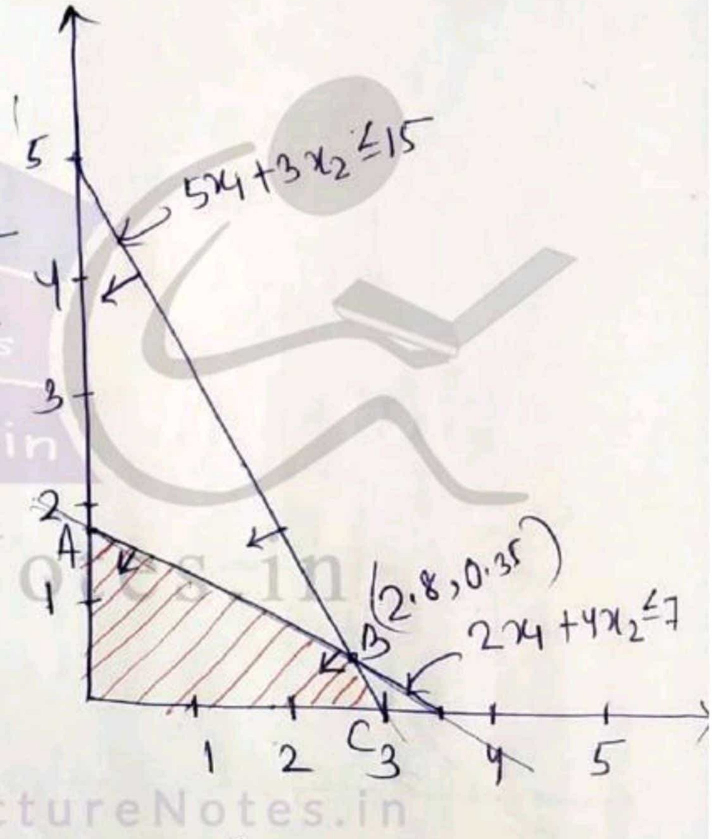
$(0, 3.5)$ & $(1.75, 0)$

from eqn (ii) -

$$5x_1 + 3x_2 = 15$$

$x_1 = 0, \; x_2 = 5$

$x_2 = 0, \; x_1 = 3$



By solving (i) & (ii) -

$$5\,[2x_1 + 4x_2 = 7] \rightarrow 10x_1 + 20x_2 = 35$$

$$2\,[5x_1 + 3x_2 = 15] \quad (-)\; 10x_1 + 6x_2 = 30$$

$$14x_2 = 5$$

$$x_2 = 5/14 = 0.35$$

$$2x_1 + 4x_2 = 7$$

$$2x_1 + 4(0.35) = 7$$

$$x_1 = 2.8$$

Point	Value of $Z = x_1 + 4x_2$
$A(0, 7/4)$	$\cancel{8}\ 7$ → Max
$B(2.8, 0.35)$	4.2
$C(3, 0)$	3

$$\text{Max } Z = x_1 + 4x_2 = 7$$

$1 \leq x \leq 2$

Max $Z = 7$, $x_1 = 0$, $x_2 = 7/4 = 1.75$

- $x_2 \leq 1$: **Subproblem-1**
  - Max $Z = 11/2$
  - $x_1 = 3/2$, $x_2 = 1$
  - $x_1 \leq 1$: **Sub problem-3**
    - Max $Z = 5$
    - $x_1 = 1$, $x_2 = 1$
  - $x_1 \geq 2$: **Sub problem-4**
    - Max $Z = 5$
    - $x_1 = 1$, $x_2 = 3/4$ →
    - $x_2 \leq 0$: **Sub problem-5**
      - Max $Z = 3$
      - $x_1 = 3$, $x_2 = 0$
    - $x_2 \geq \cancel{2}\ 1$: **Sub problem-6**
      - Infeasible
- $x_2 \geq 2$: **Sub problem-2**
  - Infeasible Solution

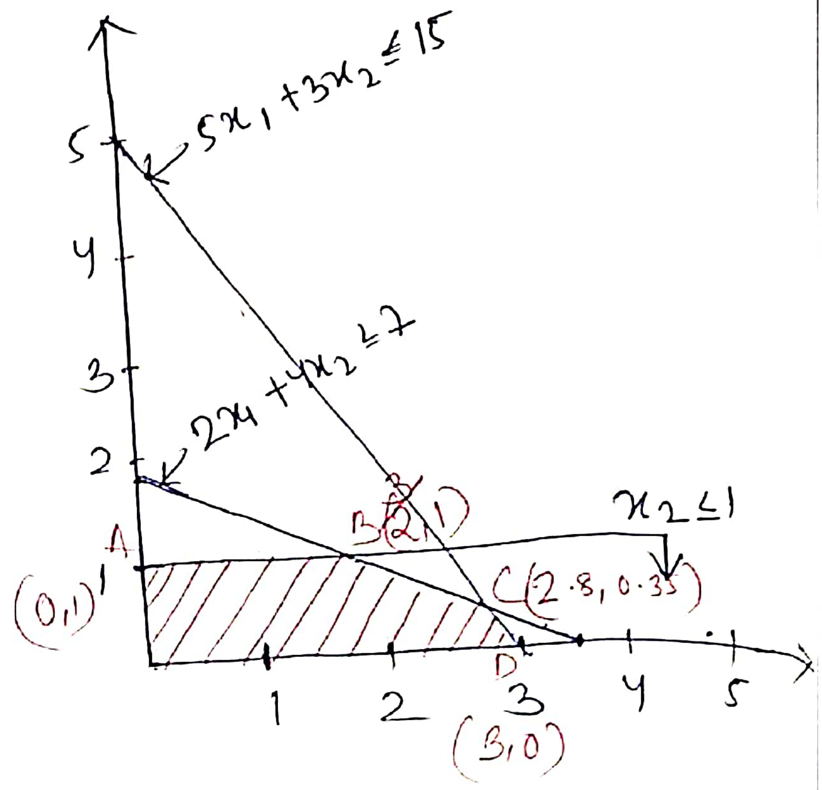
From the available integer valued sol^n the best integer solution is given by sub-problem 3

So, $x_1 = 1$, $x_2 = 1$

Max $Z = 5$

Sub problem-1

$$2x_1 + 4x_2 \le 7$$
$$5x_1 + 3x_2 \le 15$$
$$x_2 \le 1$$



$$2x_1 + 4x_2 = 7$$
$$x_2 = 1$$
$$2x_1 + 4 = 7$$
$$2x_1 = 3$$
$$x_1 = 3/2$$

1.5 $\rightarrow (3/2, 1)$

point -

$1 \le x_1 \le 2$

$x_1 \le 1, \quad x_1 \ge 2$

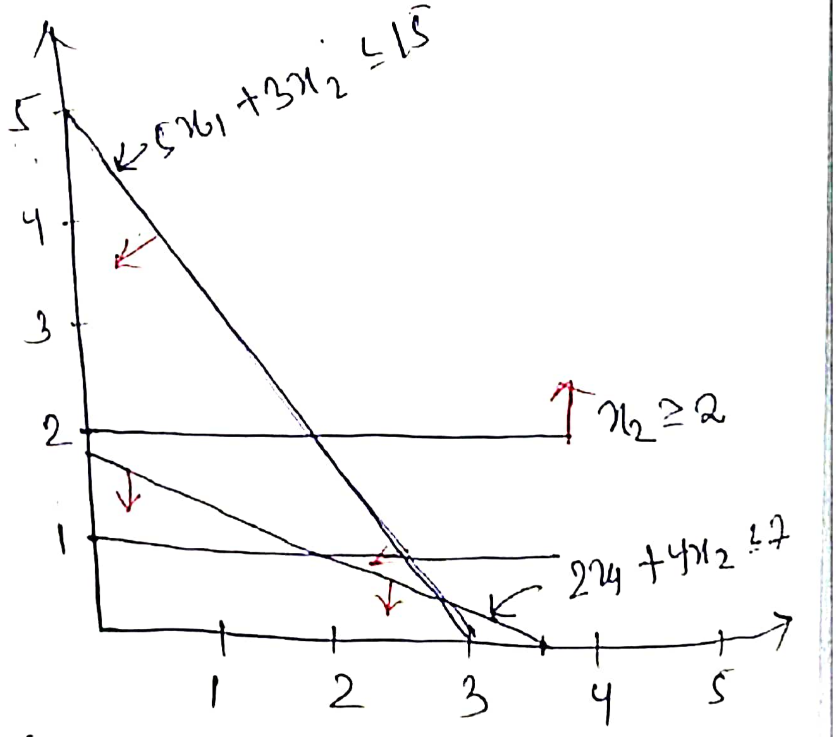
point	value of $z = x_1 + 4x_2$
A (0,1)	4
$\boxed{B\ (3/2, 1)}$	$\boxed{11/2}$ ← Max
C (2.8, 0.35)	4.2
D (3, 0)	3

Sub-problem-2

$$2x_1 + 4x_2 \le 7$$
$$5x_1 + 3x_2 \le 15$$
$$x_2 \ge 2$$

(3.5, 1.75)

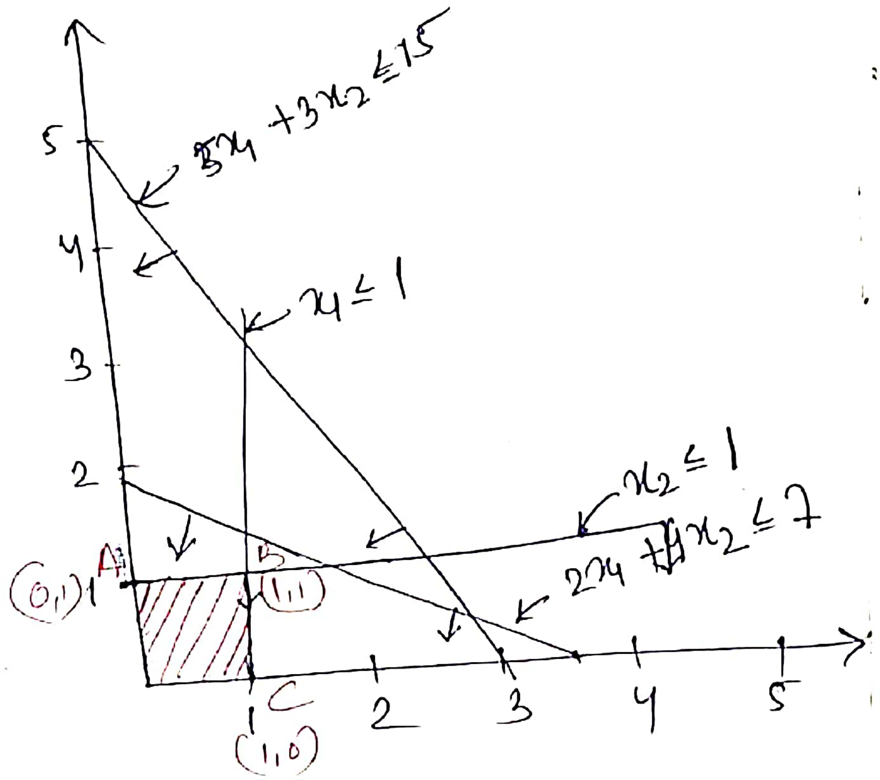
(3, 5)



No feasible region.
Infeasible Solution.

sub. problem - 3

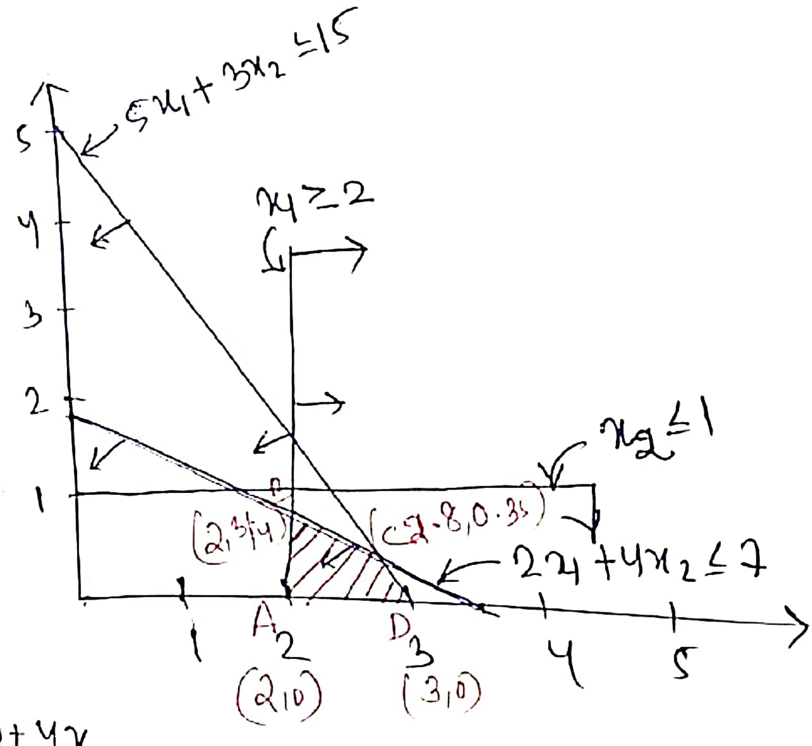
$2x_1 + 4x_2 \leq 7$
$5x_1 + 3x_2 \leq 15$
$x_2 \leq 1$
$x_1 \leq 1$



point	value of $z = x_1 + 4x_2$
A(0,1)	4
$\boxed{B(1,1)}$	$\boxed{5}$ ← Max
C(1,0)	1

sub problem - 4

$2x_1 + 4x_2 \leq 7$
$5x_1 + 3x_2 \leq 15$
$x_2 \leq 1$
$x_1 \geq 2$



$2x_1 + 4x_2 = 7$
$x_1 = 2$
$4x_2 = 3$
$x_2 = 3/4$

point	value of $z = x_1 + 4x_2$
A(2,0)	2
$\boxed{B(2, 3/4)}$	$\boxed{5}$ → Max
C(2.8, 0.35)	4.2
D(3,0)	3

Sub problem - 5

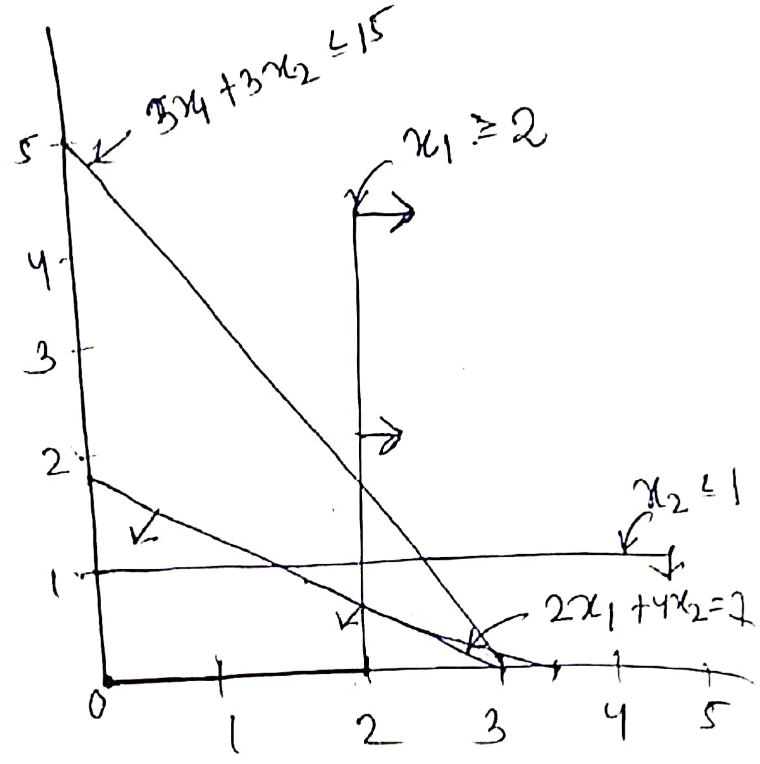
$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1 \ge 2$

$x_2 \le 0$



Max	$Z = x_1 + 4x_2$
(3, 0)	3

Sub-problem - 6

$2x_1 + 4x_2 \le 7$

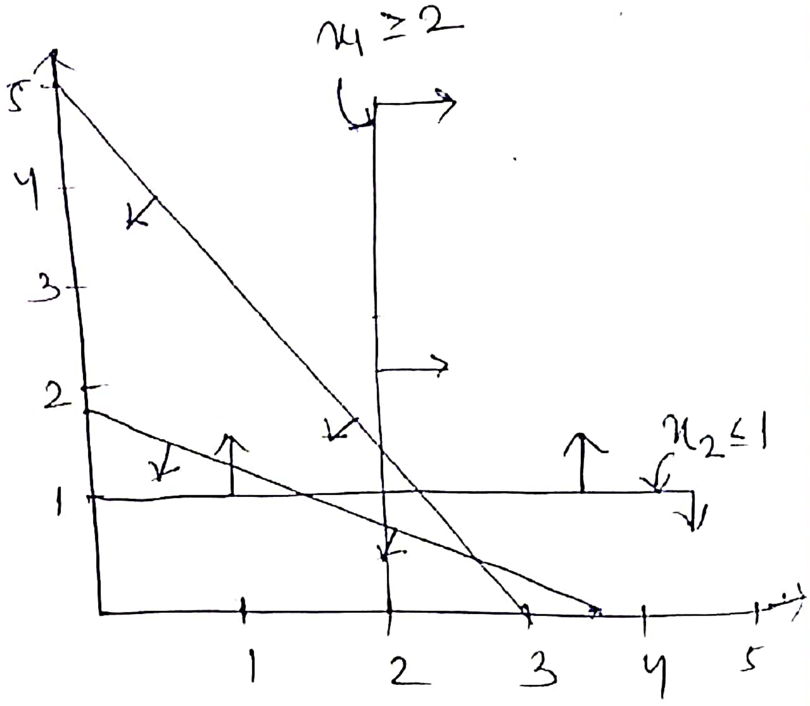
$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1 \ge 2$

$x_2 \ge 1$

No feasible region.



Module-III

Topic:

Non Linear Programming Problem

Non-Linear Programming Problem

Defn :- If the objective function / Constraints or both the objective function and Constraints are having non-linear function, then the programming having NLPP.

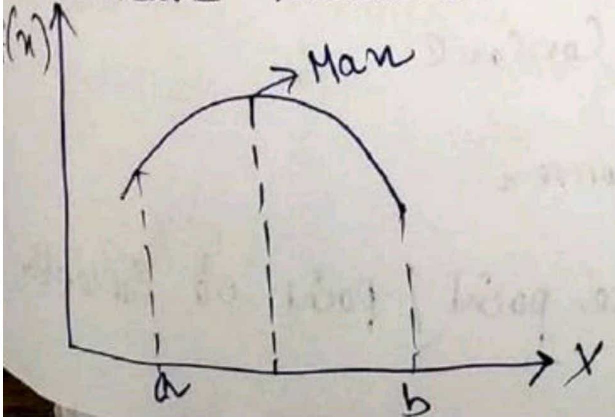
Difference b/w LPP & NLPP :-

LPP	NLPP
1) power is one.	(1) power is more than one.
2) LPP Can be Solved by simplex method.	(2) for each problem we have to apply different method.
3) Both the objective function and Constraints equation are linear.	(3) Both the O.F and Constraints are non-linear. In some case any one of them is non-linear.
$\rightarrow \text{Max } Z = CX$ $AX \geq \leq b$ $X \geq 0$	$\rightarrow Z = f(X)$ $g^n(X) \leq = \geq b$ $X \geq 0$

Concave and Convex function :-

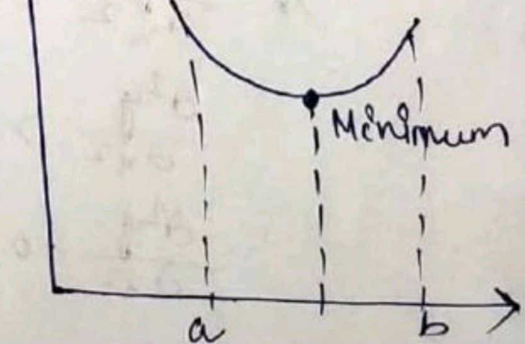
Concave function

→ A single variable function plotted in a graph always curving upward or not curving at all, is called Concave function.



Convex function

→ A single variable function plotted in a graph always curving downwards or not curving at all is called Convex function.



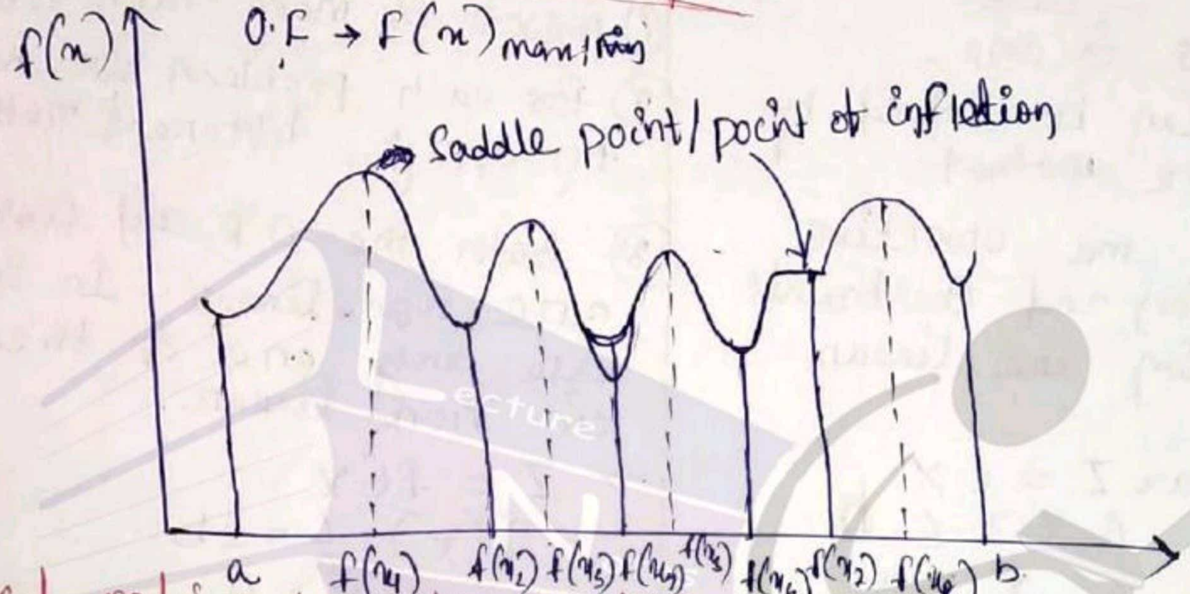
Mathematically,

If $\frac{\partial^2 f}{\partial x^2} < 0$, then the function is Concave.

If $\frac{\partial^2 f}{\partial x^2} > 0$, then the function is Convex.

Local and Global optimum :-

~~Local optima / Relative optima~~ :-



Local optima / Relative optima :-

From the graph of the non-linear programming problem, we found some Concave and Convex shape is known as "Local optima".

→ If the shape is Concave, then "local maxima" is found out. Means $f(x_1), f(x_3), f(x_5), f(x_8)$

→ If the shape is Convex, then "local minima" is found out. Means $f(x_2), f(x_4), f(x_6)$

If $\frac{\partial^2 y}{\partial x^2} < 0 \rightarrow$ Concave

$\frac{\partial^2 y}{\partial x^2} > 0 \rightarrow$ Convex

$\frac{\partial^2 y}{\partial x^2} = 0 \rightarrow$ Saddle point / point of inflection

Global optima / Absolute optima :-

The optimum value among all the values of the objective function is known as global optima.

→ Minimum of all the "local Minima" is global minimum and maxima of all the local maxima is known as global maxima.

So global value, $f(x_4) \rightarrow$ optimum value among all

* A function do not increase or decrease is known as saddle point or point of inflection.

Consider a variable function with in the interval of $x=a$ & $x=b$.

point of Maxima $= \{f(x_1), f(x_3), f(x_5), f(x_8)\}$

point of Minima $= \{f(x_2), f(x_4), f(x_6)\}$

$f(x_1) \rightarrow$ Global Maximum

$f(x_4) \rightarrow$ Global minimum

$f(x_7) \rightarrow$ Not decreasing or Not increasing

↳ Saddle point / point of inflection.

Hessian Matrix :-

A function $f(x_1, x_2, x_3 \cdots x_n)$ is called Convex function if the matrix of second order derivatives or hessian matrix is positive and the principle minor determinates of this matrix are non-negative.

$$H(x) = \frac{\partial^2 f}{}$$

$$H(x) = \begin{bmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \frac{\partial^2 F}{\partial x_1 \partial x_3} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} & \frac{\partial^2 F}{\partial x_2 \partial x_3} \\ \frac{\partial^2 F}{\partial x_3 \partial x_1} & \frac{\partial^2 F}{\partial x_3 \partial x_2} & \frac{\partial^2 F}{\partial x_3^2} \end{bmatrix}$$

If $H(x) = +ve$ (Convex function)

If $H(x) = -ve$ (Concave function)

The principle minor determinate Can be Written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}$$

* principle minor means next lower order.

$k \le m \le n$

Q) For each of the following functions show whether it is convex, concave or neither:

(a) $f(x) = 10 - x^2$

(b) $f(x) = x^4 + 6x^2 + 12x$

(c) $f(x) = 2x^3 - 3x^2$

(d) $f(x) = x^4 + x^2$

(e) $f(x) = x^3 + x^4$

(a) $f(x) = 10 - x^2$

$\frac{\partial f}{\partial x} = 0 - 2x$ and $\frac{\partial^2 f}{\partial x^2} = -2$

Since $\frac{\partial^2 f}{\partial x^2}$ is always < 0 for all values of x, the function is Concave.

(b) $f(x) = x^4 + 6x^2 + 12x$

$\frac{\partial f}{\partial x} = 4x^3 + 12x + 12$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 12$

Since $\frac{\partial^2 f}{\partial x^2}$ is always > 0 for all values of x, the function is convex.

(c) $f(x) = 2x^3 - 3x^2$

$2x - 1 = 0$
$x = 1/2$

$\frac{\partial f}{\partial x} = 6x^2 - 6x$

$\frac{\partial^2 f}{\partial x^2} = 12x - 6 = 0$

At $x = 0$, $\frac{\partial^2 f}{\partial x^2}$ is negative. (Concave)

at $x = 1/2$, $\frac{\partial^2 f}{\partial x^2} = 0$ (saddle point)

and all values of ~~x goes~~ $x > 1/2$.

$\frac{\partial^2 f}{\partial x^2}$ is +ve.

Hence the function is convex for $x \geq 1/2$.

(d) $f(x) = x^4 + x^2$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 2$

$\frac{\partial^2 f}{\partial x^2} > 0$ for all values of x, the funⁿ is convex.

(e) $f(x) = x^3 + x^4$

$\frac{\partial f}{\partial x} = 3x^2 + 4x^3$

$\frac{\partial^2 f}{\partial x^2} = 6x + 12x^2$

is +ve ≥ 0 for all values of $x \geq -0.5$. Thus the function is convex when x is ≥ 0.5

(Q) for each of the following functions determine whether it is Convex, Concave or neither.

(a) $f(x) = x_1x_2 - x_1^2 - x_2^2$

(b) $f(x) = 3x_1 + 2x_1^2 + 4x_2 + x_2^2 - 2x_1x_2$

Find out the maximum or minimum solution of the point.

(a) In this problem all the functions are two-variable functions. For the two variable ~~problem~~ funⁿ the Convexity test is as given -

Quantity	Convex	Strictly Convex	~~Strictly~~ Concave	Strictly Concave

Principle of minor determination $(-1)^k$

$(-1)^2 = (+)1$

$+ + - \rightarrow$ Convex

$- - + \rightarrow$ Con Cave

$- + - \rightarrow$ Con Cave (Take 1st sign)

$+ - + \rightarrow$ Convex (")

Ans (a) The necessary conditions are $\frac{\partial f}{\partial x_1} = 0$

and $\frac{\partial f}{\partial x_2} = 0$

$f(x) = 3x_1 + 2x_1^2 + 4x_2 + x_2^2 - 2x_1x_2$

$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 3 + 4x_1 - 2x_2 = 0$

$4x_1 - 2x_2 + 3 = 0$ —— (1)

$\frac{\partial f}{\partial x_2} = 0 \Rightarrow 4 + 2x_2 - 2x_1 = 0$

$-2x_1 + 2x_2 + 4 = 0$ —— (2)

By solving (1) & (2) -

$4x_1 - \cancel{2x_2} + 3 = 0$

$-2x_1 + \cancel{2x_2} + 4 = 0$

$2x_1 + 7 = 0$

$x_1 = -7/2$

$x_2 = -11/2$

Hessian Matrix can written as,

$$H(x) = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix}$$

$\frac{\partial^2 f}{\partial x_1^2} = 4$, $\frac{\partial^2 f}{\partial x_1 \partial x_2} = -2$

$\frac{\partial^2 f}{\partial x_2^2} = 2$ $\frac{\partial^2 f}{\partial x_2 \partial x_1} = -2$

$$H(x) = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$$

Next lower order

Determinate $H(x) = 8 - (4) = 4 = a_{11}$

The principle minor of following determinate are, $a_{11} = 4$, $a_{22} = 4$

H(x) is +ve and all the principle minor determinate are also positive so it is convex function.

As the function is convex, so it has a local minimum.

$$x_1 = -7/2$$

$$x_2 = -11/2$$

$$f(x)_{min} = 3x_1 + 2x_1^2 + 4x_2 + x_2^2 - 2x_1x_2$$

$$= 3\left(-7/2\right) + 2\left(-7/2\right)^2 + 4\left(-11/2\right) + \left(-11/2\right)^2 - 2\left(-7/2\right)\left(-11/2\right)$$

$$= \frac{-65}{4}.$$

Q) Consider the following N.L.P.P

Minimize $Z = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$

by separating this function into three one-variable functions, show that the function is convex by solving each one-variable funn by Calculus?

Soln The given function can be seperated into three functions $f(x_1)$, $f(x_2)$ and $f(x_3)$ such that

$$f(x_1) = 2x_1^2 - 24x_1$$
$$f(x_2) = 2x_2^2 - 8x_2$$
$$f(x_3) = 2x_3^2 - 12x_3 + 200$$

Now, $\frac{\partial f}{\partial x_1} = 4x_1 - 24$, $\frac{\partial^2 f}{\partial x_1^2} = 4 > 0$

function $f(x_1)$ is convex.

$$\frac{\partial f}{\partial x_2} = 4x_2 - 8 \quad \frac{\partial^2 f}{\partial x_2^2} = 4 > 0$$

Function $f(x_2)$ is convex.

$$f(x_3) = 2x_3^2 - 12x_3 + 200$$

$$\frac{\partial f}{\partial x_3} = 4x_3 - 12 : \frac{\partial^2 f}{\partial x_3^2} = 4 > 0 \text{ (Convex)}$$

Since the three component functions are convex, the given function which is the sum of these convex functions is also convex.

Now, $\frac{\partial f}{\partial x_1} = 4x_1 - 24 = 0 \Rightarrow x_1 = 6$

$$\frac{\partial f}{\partial x_2} = 4x_2 - 8 = 0 \Rightarrow x_2 = 2$$

$\frac{\partial f}{\partial x_3} = 4x_3 - 12 = 0$ gives $x_3 = 3$

Thus the optimal solution –

$$Z_{min} = 2(6)^2 - 24(6) + 2(2)^2 - 8(2) + 2(3)^2 - 12(3) + 200$$

$$= 102$$

Types of Non-Linear programming problem :–

① **Unconstraints optimization :–**

If the problem is having objective function and no constraints are there, and the objective function is non-linear then the problem is unconstraints optimization.

② **Linear Constraints optimization :–**

If the O.F is non-linear and the Constraints are linear than the problem is said to be linear constraints optimization.

③ **Separable programming :–**

A function is said separable, when it can be expressed as a sum of subfunctions where each subfunction is a function of one variable only.

Ex : $f(x)$ is an separable function

$$f(x) = f(x_1) + f(x_2) + \cdots f(x_n)$$

④ **Quadratic Programming :–** O.f → Quadratic, Constraints → Linear

If the objective function contains the term which are, either square of a variable or product of two variable.

(Q) Determine the Relative maximum or minimum of the following problem.

$$x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$$

Sol^n The necessary Conditions are -

$$\frac{\partial f_1}{\partial x_1} = 1 - 2x_1 = 0 \Rightarrow x_1 = 1/2$$

$$\frac{\partial f_1}{\partial x_2} = x_3 - 2x_2 = 0 \quad \text{— (i)}$$

$$\frac{\partial f}{\partial x_3} = 2 + x_2 - 2x_3 = 0 \quad \text{— (ii)}$$

By solving (i) & (ii) -

$$x_2 = 2/3 \quad \text{and} \quad x_3 = 4/3$$

The Sufficient Condition for determining wether the function maximise or minimize can defind as -

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x_1^2} = -2 \qquad \frac{\partial^2 f}{\partial x_2^2} = -2 \qquad \frac{\partial^2 f}{\partial x_3^2} = -2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0 \qquad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0 \qquad \frac{\partial^2 f}{\partial x_3 \partial x_1} = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_3} = 0 \qquad \frac{\partial^2 f}{\partial x_2 \partial x_3} = 1 \qquad \frac{\partial^2 f}{\partial x_3 \partial x_2} = 1$$

Topic:

Lagrange Multiplier Method

Problem with all all quality Constraints –

① Lagrange Multiplier Method –

(a) Single equality type Constraints eqn.

Let us Consider a two Variable problem with single equality type Constraint.

The problem can be formulated as -

Maximize or minimize $Z = f(x_1, x_2)$

S.T $g(x_1, x_2) = b$

and $x_1, x_2 \geq 0$

Where the objective function along with the constraints equation are differentiable with respect to x_1 and x_2.

The constraint function can be replaced by another differentiable function $h(x_1, x_2)$ such that -

$$h(x_1, x_2) = g(x_1, x_2) - b = 0$$

Now the problem can be reduced to

maximize or minimize

$Z = f(x_1, x_2)$

S.T $h(x_1, x_2) = 0$

and $x_1, x_2 \geq 0$

Lagrange function can be formulated as -

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda [h(x_1, x_2)]$$

The necessary condn of the maximise or minimize of the O.F.

$$\frac{\partial L}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial \lambda} = 0$$

The sufficient condition for determining the above solution results in maximization or minimization of the O.F involves the fun^n of $(n-1)$ of the principle minor of the following determinate having a order of -

$$D_{n+1} = \begin{bmatrix} 0 & \partial h/\partial x_1 & \partial h/\partial x_2 & \cdots & \partial h/\partial x_n \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & & & & \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

n = no. of variable.

If D_{n+1} is +ve then that is local maximum value.

D_{n+1} is -ve then that is local minimum value.

(Q) Determine the optimal solution of the following NLPP and check wether it maximize or minimize the o.f

$$\text{optimize } z = x_1^2 - 10x_1 + x_2^2 - 6x_2 + x_3^2 - 4x_3$$
$$\text{S.T } x_1 + x_2 + x_3 = 7$$
$$x_1, x_2, x_3 \geq 0$$

Soln The constraint function can be replaced by another differentiable funn i.e

$$h(x_1 + x_2 + x_3) = x_1 + x_2 + x_3 - 7 = 0$$

* Now the Lagrange function can formulated as -

$$L(x_1, x_2, x_3, \lambda) = x_1^2 - 10x_1 + x_2^2 - 6x_2 + x_3^2 - 4x_3 - \lambda(x_1 + x_2 + x_3 - 7)$$

The necessary Condition of the maximize or minimize of the O.F are -

$$\frac{\partial L}{\partial x_1} = 0 \qquad \frac{\partial L}{\partial x_3} = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \qquad \frac{\partial L}{\partial \lambda} = 0$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 10 - \lambda = 0 \qquad \frac{\partial L}{\partial x_3} = 2x_3 - 4 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 6 - \lambda = 0 \qquad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 7) = 0$$

$$2x_1 - 10 - \lambda = 0 \Rightarrow x_1 = \frac{10+\lambda}{2}$$

$$2x_2 - 6 - \lambda = 0 \Rightarrow x_2 = \frac{6+\lambda}{2}$$

$$2x_3 - 4 - \lambda = 0 \Rightarrow x_3 = \frac{4+\lambda}{2}$$

$$-(x_1 + x_2 + x_3 - 7) = 0$$

The Sufficient Condition for determining the above solution result maximization or minimization involve the situation of (3-1) of the principle minor of D_3, D_4 of the determinate can be Calculated as -

$n = 3$

$D_{n+1} = D_4$

$$D_4 = \begin{bmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix}$$

$$= \quad 0 \quad 1$$

$$D_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} = 0\times(\ ) - 1\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} + 1\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} - 1\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= -1(1\times 4) + 1\left(0 - 2(2)\right] - 1\left[1\times 0 - 2(0-2)\right]$$

$$= \underline{\underline{-12}}$$

$$D_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$D_3 = \cancel{0(1\times 0 2) + 1\times 2} - 1(1\times 2) + 1(0\times 1 - 1\times 2)$$

$$= \underline{\underline{-4}}$$

As both the D_3 and D_4 are negative so it minimize the function.

$$f(x_1, x_2, x_3)_{min} = 35$$

(1)(b) Problem with more than one equality Constraints :-

(Q) Solve the L.P.P is given below

optimize $Z = x_1^2 + x_2^2 + x_3^2$

S.T $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

and $x_1, x_2, x_3 \geq 0$

Soln The O.F and Constraint equation are differentiable w.r.t x_1, x_2, x_3

Then the Lagrange function Can be formulated as,

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

The necessary Conditions ~~are~~ for the optimal Solution Can be written as -

$$\frac{\partial L}{\partial x_1} = 0 \qquad \frac{\partial L}{\partial \lambda_1} = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \qquad \frac{\partial L}{\partial \lambda_2} = 0$$

$$\frac{\partial L}{\partial x_3} = 0$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0 \Rightarrow x_1 = \frac{\lambda_1 + 5\lambda_2}{2} \quad \text{—(I)}$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2 = 0 \Rightarrow x_2 = \frac{2\lambda_2 + \lambda_1}{2} \quad \text{—(II)}$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2 = 0 \Rightarrow x_3 = \frac{3\lambda_1 + \lambda_2}{2} \quad \text{—(III)}$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0 \quad \text{—(IV)}$$

$$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0 \quad \text{—(V)}$$

from equ^n (1) -

$$-(x_1 + x_2 + 3x_3 - 2) = 0$$

$$-\left(\frac{\lambda_1+5\lambda_2}{2} + \frac{\lambda_1 + 2\lambda_2}{2} + 3\left(\frac{3\lambda_1+\lambda_2}{2}\right) - 2\right) = 0$$

$$\Rightarrow -\left(\frac{\lambda_1+5\lambda_2+\lambda_1+2\lambda_2+9\lambda_1+3\lambda_2}{2} - 2\right) = 0$$

$$\Rightarrow -\left(\frac{11\lambda_1+10\lambda_2-4}{2}\right) = 0$$

$$\Rightarrow -11\lambda_1 - 10\lambda_2 + 4 = 0$$

$$\Rightarrow \boxed{11\lambda_1 + 10\lambda_2 - 4 = 0} \quad \text{——— (a)}$$

from equ^n (11) -

$$\left\{5\left(\frac{\lambda_1+5\lambda_2}{2}\right) + 2\left(\frac{\lambda_1+2\lambda_2}{2}\right) + \left(\frac{3\lambda_1+\lambda_2}{2}\right) - 5\right\} = 0$$

$$\Rightarrow \left(\frac{5\lambda_1+25\lambda_2+2\lambda_1+4\lambda_2+3\lambda_1+\lambda_2-10}{2}\right) = 0$$

$$\Rightarrow \boxed{10\lambda_1 + 30\lambda_2 - 10 = 0} \quad \text{——— (b)}$$

By Solving (a) & (b) -

$$\lambda_1 = 0.0869 \; \& \; \lambda_2 = 0.304$$

The Sufficient Condition for determining wether the above solution results in maximization or minimization can be obtain by -

boardered Hessian matrix

Principle of minor determinant i.e $(n-m) = (3-2) = 1$

$n =$ No. of Variable

$m =$ No. of Constraint

The determinant order $2m+1 = (2\times2)+1 = 5$

for $\max^m = (-1)^{m+n} = (-1)^5 = -ve \rightarrow$ Maximize

for $\min^m = (-1)^m = (-1)^2 = +ve \rightarrow$ minimize

$$H_B = \left[\begin{array}{c|c} 0 & P \\ \hline P^T & q \end{array}\right]$$

0 = Null Matrix $(m\times m)$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \end{bmatrix}_{m\times n}$$

Differentiate Constraint eqn with Variable

P^T = Transpose of P

$$q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$x_1 + x_2 + 3x_3 - 2 = 0$

$5x_1 + 2x_2 + x_3 - 5 = 0$

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix}, \quad P^T = \begin{bmatrix} 1 & 5 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H_B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ \hline 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{array}\right]$$

$$= 1 \begin{bmatrix} 0 & 0 & 2 & 1 \\ 1 & 5 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 0 & 2 \end{bmatrix} - 1$$

$= 460$

~~Here, $n = 3$, $m = 2$~~

~~$n - m = 3 - 2 = 1$~~

Sign of the $H(B)$ is > 0 so Convex.

Minimise

The sign of the value $H(B)$ is same that of $(-1)^m = (-1)^2 = 1$ for that objective function is minimization.

$\lambda_1 = \frac{3}{23}$, $\lambda_2 = 7/23$

$x_1 = 37/46$, $x_2 = 16/46$, $x_3 = 13/46$

$$\text{Min } z = \frac{897}{1058}$$

Topic:

Kuhn Tucker Condition

Kuhn - Tucker Condition :-

(1) (C) <u>Problem with single Inequality type Constraints</u> :-

Maximize $Z = f(x)$

$$\text{s.t. } g(x) \le b \rightarrow h(x) \le 0$$

$$x \ge 0$$

$$h(x) = g(x) - b$$

$S_1^2 \rightarrow$ To make the inequality to equality we have to add slack in non linear form.

Maximize $Z = f(x_1)$

$$\text{s.t. } h(x) + S_1^2 = 0$$

$$\text{and } x, S \ge 0$$

$$L(x_1, \lambda_1, S_1) = f(x) - \lambda \{h(x) + S_1^2\}$$

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \lambda} = 0, \quad \frac{\partial L}{\partial S} = 0$$

either $\lambda = 0$ or $S \neq 0$

or $S = 0$

For maximization problem the necessary Cond^n

$$\frac{\partial L}{\partial x} = 0$$

$$\lambda \cdot h(x) = 0$$

$$h(x) \le 0$$

$$\lambda \ge 0$$

For minimization problem

$$\frac{\partial L}{\partial x} = 0$$

$$\lambda \cdot h(x) = 0$$

$$h(x) \ge 0$$

$$\lambda \le 0$$

So all the necessary Conditions are Called Khun-Tucker Condition.

(Q) Maximize $Z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

S.T $2x_1 + x_2 \le 5$

and $x_1, x_2 \ge 0$

Here,

$$f(x) = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$$

$$h(x) = 2x_1 + x_2 - 5 \le 0$$

The Lagrange function can be formulated as-

$$L(x_1, x_2, \lambda, s) = 10x_1 + 4x_2 - 2x_1^2 - x_2^2 - \lambda(2x_1 + x_2 - 5 + s^2)$$

for maximization problem,

Khun-Tucker Conditions are -

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 10 - 4x_1 - 2\lambda = 0 \quad \text{—— (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4 - 2x_2 - \lambda = 0 \quad \text{—— (2)}$$

$$\lambda h(x) = 0 \Rightarrow \lambda(2x_1 + x_2 - 5) = 0 \quad \text{—— (3)}$$

$$h(x) \le 0 \qquad 2x_1 + x_2 - 5 \le 0 \quad \text{—— (4)}$$

$$\lambda \ge 0 \qquad \lambda \ge 0 \quad \text{—— (5)}$$

from the equ^n (3), Either $\lambda = 0$ or $2x_1 + x_2 - 5 = 0$

at $\lambda = 0$, put in equ^n (1) -

$$10 - 4x_1 = 0$$

$$x_1 = 10/4 = 2.5$$

from equ^n (2), $4 - 2x_2 = 0 \Rightarrow x_2 = 2$

As equ^n (4) is not satisfying the corresponding solution then the solution is discarded.

at $2x_1 + x_2 - 5 = 0$

$$-4x_1 - 2\lambda = -10$$

$$-0x_1 - 2x_2 - \lambda = -4$$

$$x_1 = \underline{\underline{11/6}}, \quad x_2 = \underline{\underline{4/3}}, \quad \lambda = \underline{\underline{4/3}}$$
$$\quad 1.96 \qquad\qquad 1.46 \qquad\qquad 1.07$$

As the Corresponding Solution is satisfying all the five Conditions so it is an optimal Solution.

(Q) Use the Khun-Tucker Condition to Solve the LPP.

$$\text{Max } Z = 2x_1 - x_1^2 + x_2$$
$$\text{s.t. } 2x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

Soln -
$$f(x) = 2x_1 - x_1^2 + x_2$$
$$h^1(x) = 2x_1 + 3x_2 - 6 \le 0$$
$$h^2(x) = 2x_1 + x_2 - 4 \le 0$$

$$L(x_1, x_2, \lambda_1, \lambda_2, S_1, S_2) = 2x_1 - x_1^2 + x_2 - \lambda_1(2x_1 + 3x_2 - 6 + S_1^2) - \lambda_2(2x_1 + x_2 - 4 + S_2^2)$$

for maximization problem,

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2 - 2x_1 - 2\lambda_1 - 2\lambda_2 = 0 \quad \text{--- (I)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 1 - 3\lambda_1 - \lambda_2 = 0 \quad \text{--- (II)}$$

$$\lambda_1 h^1(x) = 0 \Rightarrow \lambda_1(2x_1 + 3x_2 - 6) = 0 \quad \text{--- (III)}$$

$$\lambda_2 h^2(x) = 0 \Rightarrow \lambda_2(2x_1 + x_2 - 4) = 0 \quad \text{--- (IV)}$$

$$h^1(x) = 0 \Rightarrow 2x_1 + 3x_2 - 6 \le 0 \quad \text{--- (V)}$$

$$h^2(x) = 0 \Rightarrow 2x_1 + x_2 - 4 \le 0 \quad \text{--- (VI)}$$

$$\lambda_1 \ge 0 \Rightarrow \lambda_1 \ge 0 \quad \text{--- (VII)}$$

$$\lambda_2 \ge 0 \Rightarrow \lambda_2 \ge 0 \quad \text{--- (VIII)}$$

In order to solve there are 4 cases may arrise.

1. $\lambda_1 = 0, \quad \lambda_2 = 0$
2. $\lambda_1 \neq 0, \quad \lambda_2 = 0$
3. $\lambda_1 = 0, \quad \lambda_2 \neq 0$
4. $\lambda_1 \neq 0, \quad \lambda_2 \neq 0$

Case-1

$\lambda_1 = 0, \lambda_2 = 0$ equ^n (2) is violating so the current solution can't be an optimal solution.

Case-2 If $\lambda_1 \neq 0, \quad \lambda_2 = 0$

$$2 - 2x_1 - 2\lambda_1 = 0$$

$$1 - 3\lambda_1 = 0 \Rightarrow \lambda_1 = 1/3$$

$$2x_1 + 3x_2 - 6 = 0$$

By solving,

$$x_1 = 2/3, \; x_2 = 14/9, \; \lambda_1 = 1/3$$

$$\lambda_2 = 0$$

$$Z = 2 \times 2/3 - (2/3)^2 + 14/9 - 1/3\left(2 \times 2/3 + 3 \times 14/9 - 6\right) = 22/9$$

All the condition are satisfying so it is a feasible solution

Case-3 If $\lambda_1 = 0, \lambda_2 \neq 0$

$$2 - 2x_1 - 2\lambda_1 = 0 \quad \Rightarrow x_1 = 1$$

$$1 - \lambda_2 = 0 \Rightarrow \lambda_2 = 1$$

$$2x_1 + x_2 - 4 = 0 \Rightarrow x_2 = 2$$

As the Cond^n (5) is not satisfy so it is not a feasible solution.

Case-4 - If $\lambda_1 \neq 0, \lambda_2 \neq 0$

$$2 - 2x_1 - 2\lambda_1 - 2\lambda_2 = 0$$

$$1 - 3\lambda_1 - \lambda_2 = 0$$

$$2x_1 + 3x_2 - 6 = 0$$

$$2x_1 + x_2 - 4 = 0$$

$x_1 = 3/2$, $x_2 = 1$, $\lambda_1 = 3/4$, $\lambda_2 = -5/4$

As the condition no 8 is not satisfy so it is not a feasible solution.

Topic:

Unconstraint Optimization(Fibonacci Search Method)

Unconstraint Optimization :-

Search Method -

The methods used to find the maximum/minimum of function $f(x)$.

→ In Fibonacci sequence method, we specify the limit of accuracy and calculate the end points of the intervals using the fibonacci sequence.

Definition: The interval of uncertainity is defined as the interval in which the optimum solution is known to exist.

Consider the following successive relationship which generates an infinite series of numbers.

$$x_n = x_{n-1} + x_{n-1}$$

$$n = 1, 2, 3, \ldots$$

Define $x_0 = 0$ and $x_1 = 1$

$$\boxed{F_N = F_{N-1} + F_{N-2}}$$

Identifier	Sequence	Fibonacci Number
F_0	0	1
F_1	1	1
F_2	2	2
F_3	3	3
F_4	4	5
F_5	5	8
F_6	6	13
F_7	7	21
F_8	8	34
F_9	9	55
F_{10}	10	89

Steps :-

Step-1 : Define the end points of the search A & B.

Step-2 : Define the no. of functional Evaluation N.

Step-3 : Define the minimum resolution parameter ε.

Step-4 : Define the initial interval and first interval of uncertainity as $(B-A)$.

Step 5 : Define the second interval of uncertainity as follows –

$$L_2 = \frac{1}{F_N}\left(L_0 F_{N-1} + \varepsilon(-1)^n\right)$$

Where F_N & F_{N-1} are Fibonacci Number.

Step 6 : Locate the first two functional Evaluation.

Step 7 : Calculate $f(x_1)$ & $f(x_2)$ and eliminate the interval.

Step 8 : Use the relation $L_N = L_{N-2} - L_{N-1}$ to locate point.

(Q) Find the minimum of $f(x) = x^2 - 2x$ by fibonacci search method take the interval $0 \le x \le 1.5$ and $\varepsilon = 0.25$.

Sol.

$$f(x) = x^2 - 2x$$

$$0 \le x \le 1.5$$

$\varepsilon = 0.25$ (Minimum Resolution parameter)

Step-1 Find fibonacci Number F_n using

$$\boxed{\varepsilon = \frac{1}{F_N} \le 0.25}$$

$\boxed{N = 4}$ – No. of functional Revolution / Iteration

Step-2:

End points of the intervals are $(0, 1.5)$

Here, $A = 0$

$B = 1.5$

First interval of uncertainity $\rightarrow L_1 = B - A = 1.5 - 0 = 1.5$

Step-3:

L_2 = Second Interval of uncertainity

$$L_2 = \frac{1}{f_N}\left(L_1 f_{N-1} + \varepsilon(-1)^N\right)$$

$\varepsilon = 0.25$

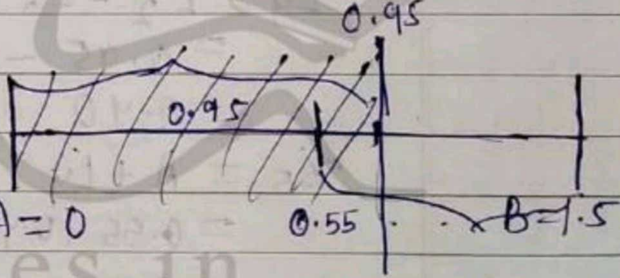
$$= \frac{1}{f_4}\left(L_1 f_3 + 0.25(-1)^4\right)$$

$$= \frac{1}{5}\left(1.5 \times 3 + 0.25(-1)^4\right)$$

$$= 0.95$$

$x_1 = A + L_2$

$= 0 + 0.95 = 0.95$



$x_2 = B - L_2$

$= 1.5 - 0.95$

$= 0.55$

$f(0.95) = (0.95)^2 - 2 \times 0.95 = -0.9975$ (Minimum)

$f(0.55) = (0.55)^2 - 2 \times 0.55 = -0.7975$ Ma

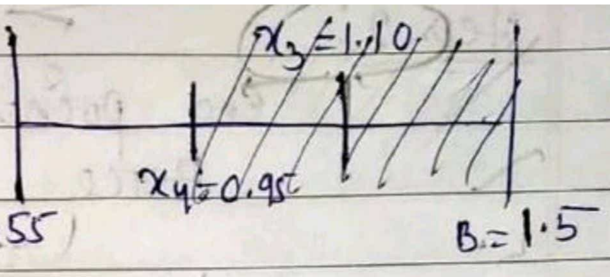
As $f(0.95) < f(0.55)$ so the optimal lies in between 0.55 to 1.5 $(0.55, 1.5)$.

Step-4:

Now $A = 0.55$

$B = 1.5$

$L_3 = L_1 - L_2$
$= 1.5 - 0.95$
$= 0.55$



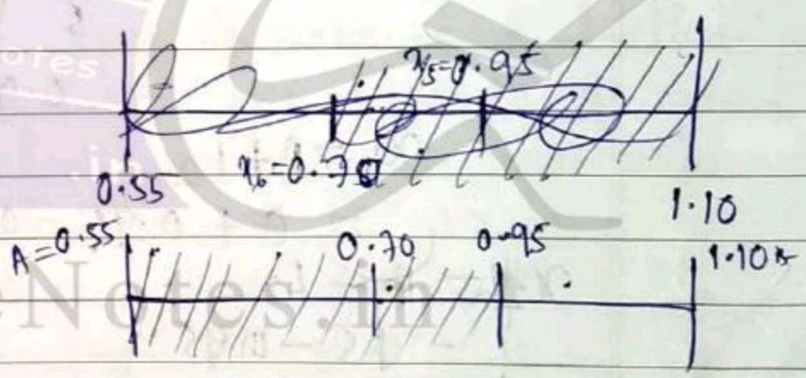
$x_3 = A + L_3 = 0.55 + 0.55 = 1.10$
$x_4 = B - L_3 = 1.5 - 0.55 = 0.95$

$f(1.10) = -0.9900$
$f(0.95) = -0.9975$ (Min)

As $f(0.95) < f(1.10)$ So the optimal lies in between $(0.55, 1.10)$

Step-5:

$A = 0.55, \ B = 1.10$
$L_4 = L_2 - L_3$
$= 0.95 - 0.55$
$= 0.40$
$x_5 = A + L_4$
$= 0.55 + 0.40$
$= 0.95$



$x_6 = B - L_4 = 1.10 - 0.40 = 0.70$
$f(0.95) = (.95)^2 - 2 \times .95 = -0.9975$ (min)
$f(0.70) = (.70)^2 - 2 \times .70 = -0.91$

As $f(0.95) < f(0.70)$
$(0.7, 1.10)$

Step-6:

$A = 0.7, \ B = 1.10$
$L_5 = L_3 - L_4 = 0.55 - 0.40 = 0.15 < \varepsilon$
The Solution is,
$\frac{0.7 + 1.10}{2} = 0.9$ (Ans)

Topic:

Golden Section Search Method

Golden Section Search :-

Modified version of Fibonacci method.

→ In Compare to the fibonacci method, the golden search is less efficient as it is derived from Fibonacci method.

$$L_2 = \frac{1}{f_N}\left(L_1 . f_{N-1} + \varepsilon(-1)^N\right)$$

$\boxed{\varepsilon = 0}$

$\boxed{N = \infty}$

$$= L_1\left(\frac{f_{N-1}}{f_N}\right) \rightarrow \text{Golden Ratio} = 0.618$$

$$x_1 = A + L_2 = A + 0.618(B-A)$$
$$x_2 = B - L_2 = B - 0.618(B-A)$$

(Q) Minimize $f(x) = x^4 - 15x^3 + 72x^2 - 1135x$

Terminate the search when

$|f(x_n) - f(x_{n-1})| \leq 0.50$

The initial range of x is $1 \leq x \leq 15$

Solution

step 1) Given $A = 1$, $B = 15$

$L_0 = B - A = 15 - 1 = 14$

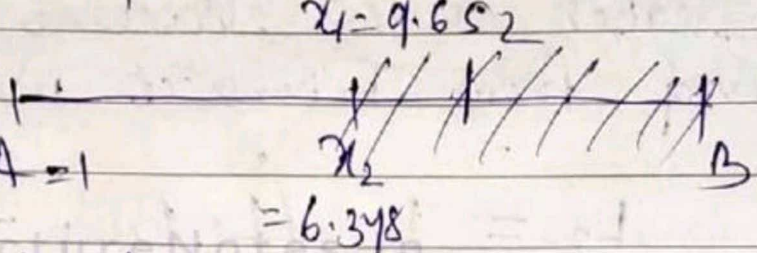
$$x_1 = A + 0.618(B-A)$$
$$= 1 + 0.618(15-1)$$
$$= 9.652$$
$$x_2 = B - 0.618(B-A)$$
$$= 15 - 0.618(14)$$
$$= 6.348$$

$$f(x_1) = (9.652)^4 - 15\times(9.652)^3 + 72(9.652)^2 - 1135\times 9.652$$
$$= 595.70$$

$f(x_2) = -168.82$

$f(x_1) > f(x_2)$

$\cancel{|f(x_2) - f(x_1)| =}$ $|f(x_2) - f(x_1)| = 764.52$



The optimal lies in b/w $1 \le x \le 9.652$.

Step-2

Now $A = 1$

$B = 9.652$

$L_2 = \cancel{x_4 - A}\ B - A = 9.652 - 1 = 8.652$ ✓

$x_3 = A + 0.618(B-A) = 1 + 0.618(8.652) = 6.346$

$x_4 = B - 0.618(B-A)$

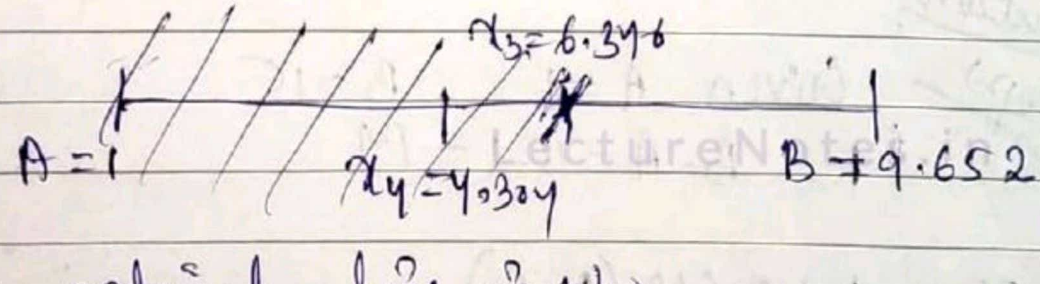
$= 9.652 - 0.618(8.652)$

$= 4.304$

$f(x_3) = -168.80 \ldots (\text{Min})$

$f(x_4) = -100.06$

$f(x_4) > f(x_3)$



The optimal lies in b/w

$4.304 \le x \le 9.652$.

$|f(x_4) - f(x_3)| = 68.74$

Step-3

$A = 4.304$

$B = 9.652$

$L_3 = B - A = 9.652 - 4.304 = 5.348$

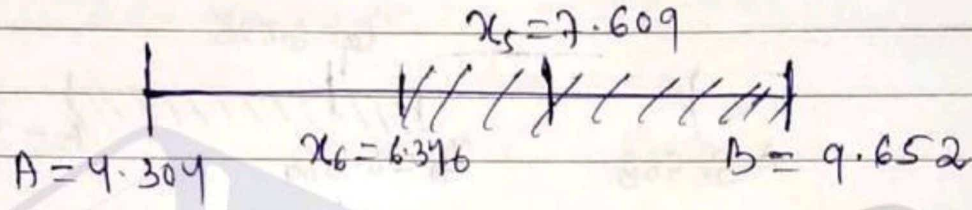
$x_5 = A + 0.618(B - A) = 7.609$

$x_6 = B - 0.618(B - A) = 6.346$

$f(x_5) = -114.64$ (Max)

$f(x_6) = -168.80$ (Min)

$f(x_6) < f(x_5)$



$4.304 \le x \le 7.609$

$|f(x_6) - f(x_5)| = 54.16$

Step-4

$A = 4.304$

$B = 7.609$

$L_4 = B - A = 7.609 - 4.304 = 3.305$

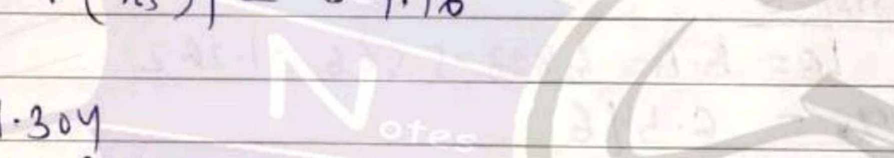
$x_7 = A + 0.618(B - A) = 6.346$

$x_8 = B - 0.618(B - A) = 5.566$

$f(x_7) = -168.80$ (Min)

$f(x_8) = -147.61$

$f(x_7) < f(x_8)$



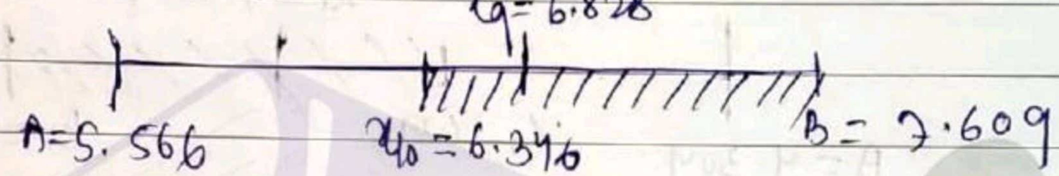
$5.566 \le x \le 7.609$

$|f(x_8) - f(x_7)| = |-147.61 + 168.80|$

$= 21.19$

Step 5

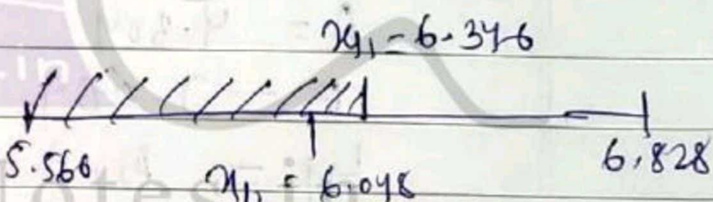
$$A = 5.566$$
$$B = 7.609$$
$$L_5 = B - A = 2.043$$
$$x_9 = A + 0.618\,(B-A) = 6.828$$
$$x_{10} = B - 0.618\,(B-A) = 6.346$$
$$f(x_9) = -166.42$$
$$f(x_{10}) = -168.80 \quad (\text{Min})$$
$$|f(x_{10}) - f(x_9)| = 2.38 \checkmark$$



$$5.566 \le x \le 6.828$$

Step 6

$$L_6 = B - A = 6.828 - 5.566 = 1.262$$
$$x_{11} = 6.346$$
$$x_{12} = 6.048$$
$$f(x_{11}) = -168.80 \quad (\text{Min})$$
$$f(x_{12}) = -163.25$$
$$|f(x_{12}) - f(x_{11})| =$$



Topic:

Quadratic Programming

Quadratic programming

Quadratic means some of the variable or square of the variable.

objective fu^n is non-linear

constraint eq^n is linear

It can be solved by two method

1) Wolf's method

2) Beal's method

<u>Wolf's method:-</u>

$$\max z = 2x_1 + 3x_2 - 2x_1^2$$

$$\text{STC} \quad x_1 + 4x_2 \le 4$$

$$x_1 + x_2 \le 2$$

$$x_1, x_2 \ge 0$$

change inequality into equality by adding slack variable

$$\max z = 2x_1 + 3x_2 - 2x_1^2$$

$$x_1 + 4x_2 + s_1^2 = 4 \qquad h_1(x) = g_1(x) - b_1 = 0$$

$$x_1 + x_2 + s_2^2 = 2 \qquad h_2(x) = g_2(x) - b_2 = 0$$

$$x_1, x_2, s_1, s_2 \ge 0$$

Now create a new constraint eq^n with respect to the variable in the form of $\boxed{-x_j + r_j^2 = 0}$ x_j = no of variable $j = 1, 2, \dots n$

$$-x_1 + r_1^2 = 0$$

$$-x_2 + r_2^2 = 0$$

formulate ~~const~~ lagrange fu^n.

$$L(x_1, x_2, s_1, s_2, r_1, r_2, \lambda_1, \lambda_2, \mu_1, \mu_2)$$

$$= 2x_1 + 3x_2 - 2x_1^2 - \lambda_1(x_1 + 4x_2 + s_1^2 - 4) - \lambda_2(x_1 + x_2 + s_2^2 - 2)$$

$$- \mu_1(-x_1 + r_1^2) - \mu_2(-x_2 + r_2^2)$$

Necessary cond^n for Lagrange func (8)

$$\frac{\partial L}{\partial x_1} = 2 - 4x_1 - \lambda_1 - \lambda_2 + \mu_1 = 0 \quad \text{(I)}$$

$$\frac{\partial L}{\partial x_2} = 3 - 4\lambda_1 - \lambda_2 + \mu_2 = 0 \quad \text{(II)}$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0 \quad \text{(III)}$$

$$\frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0 \quad \text{(IV)}$$

$$\frac{\partial L}{\partial r_1} = -2\mu_1 r_1 = 0 \quad \text{(V)}$$

$$\frac{\partial L}{\partial r_2} = -2\mu_2 r_2 = 0 \quad \text{(VI)}$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + 4x_2 + s_1^2 - 4) = 0 \quad \text{(VII)}$$

$$\frac{\partial L}{\partial \lambda_2} = -(x_1 + x_2 + s_2^2 - 2) = 0 \quad \text{(VIII)}$$

$$\frac{\partial L}{\partial \mu_1} = -(-x_1 + r_1^2) = 0 \quad \text{(IX)}$$

$$\frac{\partial L}{\partial \mu_2} = -(-x_2 + r_2^2) = 0 \quad \text{(X)}$$

Now simplify equ^n (III), (IV), (V), (VI) we will get -

$$-2\lambda_1 s_1 = -2\lambda_2 s_2 = 0$$

$$\therefore \lambda_1 s_1 = \lambda_2 s_2 = 0$$

from (V) & (VI) $\mu_1 r_1 = \mu_2 r_2 = 0$

$$\therefore \boxed{\begin{matrix} \lambda_1 s_1 = \lambda_2 s_2 = 0 \\ \mu_1 r_1 = \mu_2 r_2 = 0 \end{matrix}}$$

The above Condition is called Complimentary Slackness Condition.

The new Complimentary slackness Condition is -

$$\mu_1 x_1 = \mu_2 x_2 = 0$$

[from equ^n (ix) & (x) We will get

$x_1 = r_1^2$, $x_2 = r_2^2$ } NLPP $\quad r_1^2 = r_1$, $r_2^2 = r_2$

According to LPP $x_1 = r_1$ & $x_2 = r_2$]

The new Complimentary slackness condition is -

$$\lambda_1 s_1 = \lambda_2 s_2 = 0$$

$$\mu_1 x_1 = \mu_2 x_2 = 0$$

Now simplify equ^n (i), (ii), (vii), (viii) -

from equ^n (i)

$$4x_1 + \lambda_1 + \lambda_2 - \mu_1 = 2 \quad \text{--- (1)}$$

from (ii) $\quad 4\lambda_1 + \lambda_2 - \mu_1 = 3 \quad$ --- (2)

from (vii) $\quad x_1 + 4x_2 + s_1^2 = 4 \quad$ --- (3)

from (viii) $\quad x_1 + x_2 + s_2^2 = 2 \quad$ --- (4)

step-2

Now N.L.PP Convert the problem into L.P.P

$$4x_1 + \lambda_1 + \lambda_2 - \mu_1 = 2$$

$$4\lambda_1 + \lambda_2 - \mu_1 = 3$$

$$x_1 + 4x_2 + s_1 = 4$$

$$x_1 + x_2 + s_2 = 2$$

Now introduce artificial variable in the first two Constraint equ^n -

$$4x_1 + \lambda_1 + \lambda_2 - \mu_1 + A_1 = 2$$

$$4\lambda_1 + \lambda_2 - \mu_1 + A_2 = 3$$

$$x_1 + 4x_2 + s_1 = 4$$

$$x_1 + x_2 + s_2 = 2$$

$$x_1, x_2, \lambda_1, \lambda_2, A_1, A_2, s_1, s_2, \mu_1, \mu_2 \geq 0$$

Apply two Phase Method -

Max $z^* = ~~-A_1+A_2~~ \; 0x_1 + 0x_2 - 0 \cdot x_1^2 - A_1 - A_2 + 0S_1 + 0S_2$

$$\text{Max } z^* = -A_1 - A_2$$

C_B	B	X_B	x_1	x_2	S_1	S_2	λ_1	λ_2	μ_1	μ_2	A_1	A_2	Min Ratio X_B/X_K
	C_j		0	0	0	0	0	0	0	0	-1	-1	
-1	A_1	2	$\boxed{4}$	0	0	0	1	1	-1	0	1	0	1/2
-1	A_2	3	0	0	0	0	4	1	0	-1	0	1	–
0	S_1	4	1	4	1	0	0	0	0	0	0	0	4
0	S_2	2	1	1	0	1	0	0	0	0	0	0	2
	$Z_j - C_j$	-5	↑-4	0	0	0	-5	-2	1	1	0	0	

Because of Complimentary Slackness Condition λ_1, λ_2 Cannot enter the bases.

So Enter x_1 and A_1 will leave the bases.

C_B	B	X_B	x_1	x_2	S_1	S_2	λ_1	λ_2	μ_1	μ_2	A_1	A_2	Min Ratio
	C_j		0	0	0	0	0	0	0	0	-1	-1	
0	x_1	1/2	1	0	0	0	1/4	1/4	-1/4	0	X	0	–
-1	A_2	3	0	0	0	0	4	1	0	-1	X	1	–
0	S_1	7/2	0	$\boxed{4}$	1	0	-1/4	-1/4	1/4	0	X	0	7/8
0	S_2	3/2	0	1	0	1	-1/4	-1/4	1/4	0	X	0	3/2
	$Z_j - C_j$	-3	0	↑0	0	0	-4	-1	0	1	X	0	

Because of Complimentary Slackness Condition λ_1, λ_2 will not enter ~~so~~ the next min. value will be zero.

Now enter x_2.

C_B	B	X_B	x_1	x_2	S_1	S_2	λ_1	λ_2	μ_1	μ_2	A_1	A_2	Min Ratio
	C_j		0	0	0	0	0	0	0	0	-1	-1	
0	x_1	1/2	1	0	0	0	1/4	1/4	-1/4	0	X	0	2
-1	A_2	3	0	0	0	0	$\boxed{4}$	1	0	-1	X	1	3/4
0	x_2	7/8	0	1	1/4	0	-1/16	-1/16	1/16	0	X	0	–
0	S_2	5/8	0	0	-1/4	1	-3/16	-3/16	3/16	0	X	0	–
	$Z_j - C_j$	-3	0	0	0	0	↑-4	-1	0	1	X	0	

Now λ_1 will enter and A_2 will leave the basis.

		C_j	0	0	0	0	0	0	0	0	-1	-1
C_B	B	X_B	x_1	x_2	S_1	S_2	λ_1	λ_2	μ_1	μ_2	A_1	A_2
0	x_1	5/16	1	0	0	0	0	3/16	-1/4	1/16	x	x
0	λ_1	3/4	0	0	0	0	1	1/4	0	-1/4	x	x
0	x_2	59/64	0	1	1/4	0	0	-3/64	1/16	-1/64	x	x
0	S_2	49/64	0	0	-1/4	1	0	-9/64	3/16	-3/64	x	y
Z_j		0	0	0	0	0	0	0	0	0	-	-
$Z_j - C_j$		0	0	0	0	0	0	0	0	0	-	-

The optimum solution is –

$$x_1 = 5/16, \quad x_2 = 59/64$$

$$\text{Max } Z = 2\left(5/16\right) + 3\left(59/64\right) - 2\left(5/16\right)^2$$
$$= 3.19$$

(Q) Use the Wolfe's method to solve the QPP.

$$\text{Maximize } Z = 2x_1 + x_2 - x_1^2$$

S.T
$$2x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 + S_2^2 = 4$$
$$x_1, x_2 \ge 0$$

Topic:

Project Gradient

Project Gradient

Search Method -

It is used to find the extreme point of an objective function.

→ Linear Constraint optimization (object
Objective funⁿ - Non-linear
Constraint Equⁿ - Linear

(Q) $f(x) = (x_1 - 3)^2 + (x_2 - 4)^2$

S.T $2x_1 + x_2 = 3$

$x_1, x_2 \geq 0$

Step size $S = 2$

Let us take the initial interval $(1,1)$.
Differentiate the objective funⁿ by taking the limit $(1,1)$

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_1=1} = 2(x_1 - 3) = 2(1-3) = -4$$

$$\left.\frac{\partial f}{\partial x_2}\right|_{x_2=1} = 2(x_2 - 4) = 2(1-4) = -6$$

Now differentiate the Constraint equation

$$\frac{\partial g}{\partial x_1} = 2$$

$$\frac{\partial g}{\partial x_2} = 1$$

Now find out the value of λ_i by taking the equation —

$n = 1, 2, \cdots, j$
$m = 1, 2, \cdots, i$

Variable   Constraint

$$\sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i} \cdot \frac{\partial f}{\partial x_i} = -\sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i} \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i}$$

$$\Rightarrow \frac{\partial g}{\partial x_1} \cdot \frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_2} \cdot \frac{\partial f}{\partial x_2} = -\left[\frac{\partial g}{\partial x_1} \cdot \lambda_1 \frac{\partial g}{\partial x_1} + \frac{\partial g}{\partial x_2} \cdot \lambda_1 \cdot \frac{\partial g}{\partial x_2}\right]$$

$$\Rightarrow 2 \cdot (-4) + 1 \times (-6) = -\left[2 \cdot \lambda_1 \cdot 2 + 1 \cdot \lambda_1 \cdot 1\right]$$

$$\Rightarrow -8 - 6 = -5\lambda_1$$

$$\Rightarrow \lambda_1 = 14/5$$

Now find out the value of -

$$2\lambda_0 = \pm \sqrt{\sum_{i=1}^{n} \left[\left(\frac{\partial f}{\partial x_i}\right)^2 + \frac{\partial f}{\partial x_i} \sum_{j=1}^{m} \lambda_j \left(\frac{\partial g_j}{\partial x_j}\right)\right]^2}$$

$$= \pm \sqrt{\left[(-4)^2 + (-4) \cdot 14/5 \times 2\right]^2 + \left[(-6)^2 + (-6) \cdot \frac{14}{5} \times 1\right]^2}$$

$$2\lambda_0 = \pm 20 \cdot 24$$

$$= -20.24 \checkmark \text{ (Min)}$$

$\boxed{\begin{matrix} + \text{ Max} \\ - \text{ Min} \end{matrix}}$

Note

If $2\lambda_0 = 0$ then method will be stop - cannot proceed for next step.

By taking $(1,1)$ in objective function

$\boxed{f(x) = 13}$ ✓

Now find out the value of x_1, x_2 by taking

$$x_i^{\delta+1} = x_i^{\delta} \pm S\left[\frac{1}{2\lambda_0}\left\{\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i}\right\}\right]$$

$\delta \rightarrow$ Interval limit $(0, 1, 2, 3)$

$\delta = 0$

$$x_1^{0+1} = x_1^0 + S\left[\frac{1}{2\lambda_0}\left\{\frac{\partial f}{\partial x_1} + \sum \lambda \frac{\partial g}{\partial x_1}\right\}\right]$$

$$x_1 = 1 + 2\left[\frac{1}{-20.24}\left\{-4 + \frac{14}{5} \cdot 2\right\}\right]$$

$$= 0.843$$

$\delta = 0$, $$x_2^{\delta+1} = x_2^0 + S\left[\frac{1}{2\lambda_0}\left\{\frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2}\right\}\right]$$

$$x_2 = 1 + 2\left[\frac{1}{-20.24}\left\{(-6) + \frac{14}{5} \cdot 1\right\}\right]$$

$$= 1.314$$

put x_1 and x_2 in objective fun^n.

$$f(x) = (0.843 - 3)^2 + (1.314 - 4)^2$$

$= \cancel{.4547}$

$= \cancel{(-2.237)^2 + (-2.526)^2}$

$= \cancel{5.004 + 6.380}$

$= 11.3846$ Ans

Now put x_1 & x_2 by differentiate

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_1 = 0.763} = 2(x_1 - 3) = -4.474$$

$$\left.\frac{\partial f}{\partial x_2}\right|_{x_2 = 1.474} = 2(x_2 - 4) = -5.052$$

$$\frac{\partial g}{\partial x_1} = 2 \quad , \quad \frac{\partial g}{\partial x_2} = 1$$

$$\Rightarrow \frac{\partial g}{\partial x_1}\cdot\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_2}\cdot\frac{\partial f}{\partial x_2} = -\left[\frac{\partial g}{\partial x_1}\cdot\lambda_1\cdot\frac{\partial g}{\partial x_1} + \frac{\partial g}{\partial x_2}\cdot\lambda_1\cdot\frac{\partial g}{\partial x_2}\right]$$

$$\Rightarrow 2\cdot(-4.474) + 1\cdot(-5.052) = -\left[2\cdot\cancel{\tfrac{14}{5}}\,\lambda_1\cdot 2 + 1\cdot\lambda_1\cdot 1\right]$$

$$\Rightarrow \cancel{\lambda}$$

$$\Rightarrow -8.948 - 5.052 = -5\lambda_1$$

$$\Rightarrow -14 = -5\lambda_1$$

$$\Rightarrow \lambda_1 = 14/5$$

Now,

$$2\lambda_0 = \pm\sqrt{\sum_{i=1}^{n}\left[\left(\frac{\partial f}{\partial x_i}\right)^2 + \frac{\partial f}{\partial x_i}\sum_{j=1}^{m}\lambda_j\left(\frac{\partial g_i}{\partial x_j}\right)\right]^2}$$

$$2\lambda_0 = \pm\sqrt{\left[(-4.474)^2 + (-4.474)\cdot\frac{14}{5}\cdot 2\right]^2 + \left[(-5.052)^2 + (-5.052)\cdot\frac{14}{5}\cdot 1\right]^2}$$

$$= \pm\sqrt{(20.016 - 25.05)^2 + (25.522 - 14.145)^2}$$

$$= \sqrt{(-5.034)^2 + (11.377)^2}$$

$$= \pm\sqrt{25.341 + 129.436}$$

$$= \pm\sqrt{154.777}$$

$$= \pm 12.94$$

$$-12.94$$

Now find out the value of x_1, x_2 by taking:

$$x_i^{s+1} = x_i^s + S\left[\frac{1}{2\lambda_0}\left\{\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_i}{\partial x_j}\right\}\right]$$

$S \to$ Interval limit $(0,1,2,3)$

$s = 0$

$$x_1^{0+1} = x_1^0 + S\left[\frac{1}{2\lambda_0}\left\{\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1}\right\}\right]$$

$$x_1 = 0.763 + 2\left[\frac{1}{-12.44}\left\{(-4.474) + \frac{14}{5}\cdot 2\right]$$

$$= \cancel{12.68} \quad 0.66.$$

$$x_2^{0+1} = x_2^0 + S\left[\frac{1}{2\lambda_0}\left\{\frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2}\right\}\right]$$

$$= 1.474 + 2\left[\frac{1}{-12.44}\left\{(-5.052) + \frac{14}{5}(1)\right\}\right]$$

$$= \cancel{7.88} \quad 1.655$$

put x_1 and x_2 in objective function

$$f(x) = \cancel{(12.68-3)^2 + (7.88-4)^2}$$

$$= (0.66-3)^2 + (1.655-4)^2$$

$$= \cancel{93.70 + 15.1026}$$

$$= \cancel{108.80} \quad 10.90$$

The above is not satisfying the objective function. Go to the previous step

$x_1 = 0.763$

$x_2 = 1.474$

$f(x) = 11.3846$

Topic:

Genetic Algorithm

Genetic Algorithm :-

It is a method for solving both constrained and unconstrained optimization problems that is based on natural selection.

→ It modifies a population of individual solutions.

→ It can apply to solve a variety of optimization problems that are not well suited for standard optimization algorithm, including problems in which the O.F is discontinuous, non-differentiable or highly non-linear.

Classical Algorithm	Genetic Algorithm
→ Generates a single point in each iteration.	→ Generates a population of points at each iteration.
→ The sequence of points approaches an optimal solution.	→ The best point in the population approaches an optimal solution.

Topic:
Sensitivity Analysis

Sensitivity Analysis :- [CO1, PO1]

: Sensitivity analysis is also Called as post optimality test. This method is used for changes in the market for the diff. variables. In that case we can get profit otherwise we can get loss.

: To overcome this situation we are using the sensitivity analysis. It's also called as post optimality test because 1st we are finding out the optimum soln then introducing the new variable, then again we are finding the new optimal soln.

* once the optimum soln to be lpp is obtained 2 situations may arise;

i) During the formulation it's assumed that the parameter such as market demand resource consumption, resource availability are known with certainity & don't change over the time. But in actual practice the market fluctuates, material & labour cost goes up & down, so this parameters are varies time to time. So it's desirable to study how the current optimum soln changes when the parameter of the problem get changed.

ii) The 2nd situation is after obtaining the optimum soln one may discover that a wrong value of cost co-efficient was used or a particular variable or constraint was ommited or one or more right hand side constraints of the constraints we used wrong.

Parameter for Sensitivity analysis :-

i) changes in righthand side values of the constraints.

ii) Addition of variable

iii) Deletion of variable

(i) changes in right hand side values of the constraint : [CO1, PO2]

Q→ Max $Z = 5x_1 + 12x_2 + 4x_3$

subjected to.

$x_1 + 2x_2 + x_3 \le 5$ & $x_1, x_2, x_3 \ge 0$

$2x_1 - x_2 + 3x_3 = 2$

Discuss the effect of changing the requirement vector from $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ on the optimum soln.

Soln,

		C_j	5	12	4	0	-M
B	C_B	X_B	x_1	x_2	x_3	S_1	A_1
x_2	12	8/5	0	1	-1/5	2/5	-1/5
x_1	5	9/5	1	0	7/5	1/5	2/5

$x_1 = 9/5$, $x_2 = 8/5$

$\text{Max } z = \frac{141}{5}$

As the right hand side constant value is changed from $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$, then

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12/5 \\ 11/5 \end{bmatrix}$$

$x_1 = 11/5$, $x_2 = 12/5$

$\text{Max } z = 199/5$

ii) <u>Deletion of variable :-</u> [CO₁, PO₂]

Q→ $\max Z = 45x_1 + 100x_2 + 30x_3 + 50x_4$

subjected to

$7x_1 + 10x_2 + 4x_3 + 9x_4 \le 1200$

$3x_1 + 40x_2 + x_3 + x_4 \le 800$

& $x_1, x_2, x_3, x_4 \ge 0$.

If the product 'B' is not to be produce, so that the variable 'x_2' is to be deleted from the table. Findout the optimum soln to the resulting LPP.

soln,

	C_j		45	100	30	50	0	0
B	C_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2
x_3	-30	800/3	5/3	0	1	7/3	4/15	-1/15
x_2	100	40/3	1/30	1	0	-1/30	-1/150	2/75
			25/3	0	0	50/3	22/3	2/3

$x_1 = 0,\ x_2 = 40/3,\ x_3 = 800/3,\ x_4 = 0$

$\max Z = 9333.33$

∴ As x_2 is to be deleted from this table so we assign a largest panelty value i.e → '-M' to the variable x_2. So the simplex table can be,

B	C_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	θ_{min}
x_3	-30	800/3	5/3	0	1	7/3	4/15	-1/15	
x_2	-M	40/3	1/30	1	0	-1/30	-1/150	2/75	
			$-\frac{M}{30}+5$	0	0	$-\frac{M}{30}+\frac{60}{3}$	$8+\frac{M}{160}$	(-ve) ↑	ΔJ
x_3	30	900/3	21/12	5/2	1	27/12	75/300	0	
S_2	0	1500	5/4	75/2	0	-75/60	-75/300	1	
			7.5	M+75	0	35/2	15/2	0	

$x_1 = 0,\ x_2 = 0,\ x_3 = 900/3,\ x_4 = 0$

$\max Z = 9000$ (Ans)

Addition of variable

Q→ Max. $z = 45x_1 + 100x_2 + 30x_3 + 50x_4$

S.T

$7x_1 + 10x_2 + 4x_3 + 9x_4 \le 1200$

$3x_1 + 40x_2 + x_3 + x_4 \le 800$

& $x_1, x_2, x_3, x_4 \ge 0$.

[CO1, PO2]

If a new variable x_5 is added to this problem with a column $\begin{bmatrix}10\\10\end{bmatrix}$ & $C_5 = 120$. Find the change in the optimal soln.

Soln:

	C_j		45	100	30	50	0	0
B	C_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2
x_3	30	$\frac{800}{3}$	5/3	0	1	7/3	4/15	-1/15
x_2	100	40/3	1/30	1	0	-1/30	-1/150	2/75
			25/3	0	0	50/3	22/5	2/3

$\therefore x_1 = 0,\ x_2 = \frac{40}{3},\ x_3 = \frac{800}{3},\ x_4 = 0$

If a new variable x_5 is added to this problem with a column $\begin{bmatrix}10\\10\end{bmatrix}$ & $C_5 = 120$.

$$x_5 = \begin{bmatrix} 4/15 & -1/15 \\ -1/150 & 2/75 \end{bmatrix}\begin{bmatrix}10\\10\end{bmatrix}$$

$$= \begin{bmatrix}2\\1/5\end{bmatrix}$$

	C_j		45	100	30	50	120	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	S_1	S_2	x_B/x_k
x_3	30	800/3	5/3	0	1	7/3	2	4/15	-1/15	400/3
x_2	100	40/3	1/30	1	0	-1/30	$\boxed{1/5}$	-1/150	2/75	$\frac{200}{3}$ →
			25/3	0	0	50/3	-40	22/3	2/3	

B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	S_1	S_2
x_3	30	400/3	4/3	-10	1	8/3	0	1/3	-1/3
x_5	120	200/3	1/5	5	0	-1/6	1	-1/30	2/15
			15	200	0	10	0	6	6

$\therefore x_3 = \frac{400}{3}, x_5 = 200/3$

$\therefore Z = 12,000$.

——— END ———

Module-IV

Topic:

Queueing Theory

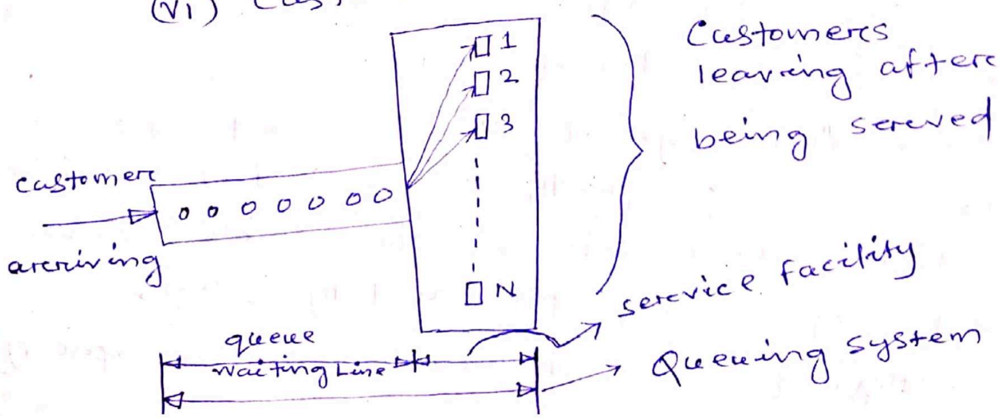
→ A flow of customers from finite or infinite population towards the service facility forms a queue. On account of lack of capability to solve them all at a time.

Constituents of an Queuing System:-

→ The major constituents of a queuing system are as follows;

1. **Customer:-** The arriving unit that being services is known as customer. The customer may be person, machine, etc.
2. **Queuing (Waiting Line) :-** The number of customer waiting to get the desire services is known as Queuing. The queue does not include the customer(s) currently being served.
3. **Service Channels/Facility/Server:-** The process or facility which provides the required service to the customer is known as service channel or facility or server. This can be single or multi-channel.
4. **Elements of a Queuing System:-** A Queuing system is specified completely by following elements.

(i) I/P or Arrival Pattern
(ii) O/P or service pattern
(iii) Service channels
(iv) Capability of the system
(v) Service disciplines
(vi) Customer's behaviour



5. Traffic Intensity (or Utilization Factor):-

→ Traffic intensity is a measurement at which a server is busy with service.

$$\text{Traffic intensity, } \rho = \frac{\text{Mean arrival rate}}{\text{Mean service rate}} = \frac{\lambda}{\mu}$$

Unit:- Erlang

6. Kendall's Notation For Representating Queuing Model:-

→ General Kendall's Notation is specifies a queuing model in symbolic form, ie

$$(a/b/c):(d/e)$$

where
a = Probability law according to which customers arrive
b = Probability law according to which customers are being served
c = No. of service channel
d = Capacity of the system
e = Queue discipline

Types of Queuing Model:- The following are the important types of Queuing model.

a. MODEL-I (M/M/1):(∞/FCFS):-

→ This represents the arrival of customers is according to poison's distribution with inter-arrival negative exponential distribution, service to the customer is according to the exponential, single server, infinity capacity with first come first serve (FCFS) queue discipline.

→ Here the symbol 'M' is used due to Markovian property of random arrival and random service to which Poison and exponential probability obey.

→ Shortly:- [Single server(s) & infinity capacity]

b. **MODEL-II** (M/M/S) : (∞/FCFS) :-

→ This represents the arrival of customers is according to Poison's distribution with inter-arrival negative exponential distribution, service to the customer is according to the exponential, multi-server, infinity capacity with first come first serve (FCFS) queue discipline.

→ Shortly it is denoted as;

[Multi-server(s) & Infinity capacity]

c. **MODEL-III** (M/M/1) : (N/FCFS)

→ This represents the arrival of customers is according to Poison's distribution with inter-arrival negative exponential distribution, service to the customer is according to the exponential, single server, the capacity of the system is limited (finite) to 'N' number of units or customers.

→ It is shortly denoted as;

[Single server(s) & finite capacity]

d. **MODEL-IV** (M/M/S) : (N/FCFS) :-

→ This represents the arrival of customers is according to Poison's distribution with inter-arrival negative exponential distribution, service to the customer is according to the exponential, multi-server (s), the capacity of the system is limited (finite) to 'N' number of units or customers.

MODEL-I (M/M/1) : (∞/FCFS) :-

[Single server & Infinity capacity]

→ Traffic intensity, $\boxed{\rho = \frac{\lambda}{\mu}}$

→ Expected (average) number of units in the system or, length of the system (L_s),

$$\boxed{L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}}$$

Scanned with CamScanner

Imp. Formulae of MODEL-I

* If the arrivals are ~~completely random~~ in a given time, the Poisson probability distribution is used.

→ Arrival is Poisson distribution with mean 'λt', $\left[P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}\right]_{n \geq 0}$

→ Service is Exponential distribution with mean $\frac{1}{\mu}$ $\left[s(t) = \begin{cases} \mu e^{-\mu t}, & t > 0 \\ 0, & t < 0 \end{cases}\right]$

→ The intervals between the successive arrivals are distributed negative exponential.

→ Probability of the system for which the system being idle $= 1 - \frac{\lambda}{\mu}$

→ Probability that a customer will have to wait $= \frac{\lambda}{\mu}$

2021/4/

Model-I : (M/M/1) : (∞/FCFS)

[Single Server, Infinity Customers]

→ Traffic Intensity, $\rho = \frac{\lambda}{\mu}$

→ Expected (average) number of units in the System or length of the System, $L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

→ Expected (Mean) Queue length, $L_q = L_s - \rho = \frac{\lambda^2}{\mu(\mu-\lambda)}$

or, $L_q = \frac{\rho^2}{1-\rho}$

→ Expected time spent in the System, $t_s = \frac{1}{\mu - \lambda} = \frac{L_s}{\lambda}$

→ Expected time spent in the Queue, $t_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{L_q}{\lambda}$

→ Probability of no units in the system, $P_0 = 1 - \rho = 1 - \frac{\lambda}{\mu}$

→ Probability of 'n' units in the system, $P_n = \rho^n(1-\rho)$

→ Probability of queue size that exceeds 'N' (i.e. $\geq N$) $= \rho^N$

→ Probability that no arrival will occur during the next 't' time, $P_0 = \frac{(\lambda t)^n . e^{-\lambda t}}{n!} = e^{-\lambda t}$ $(\because n = 0)$

Relation between 't_s' and 't_q'

$$t_s = \frac{L_s}{\lambda} = \frac{L_q + \rho}{\lambda} = \frac{L_q}{\lambda} + \frac{\rho}{\lambda}$$

$t>0$
$t<0$

$$= \frac{L_q}{\lambda} + \frac{\lambda}{\mu \times \lambda} = \frac{L_q}{\lambda} + \frac{1}{\mu}$$

$$\Rightarrow \boxed{t_s = t_q + \frac{1}{\mu}} \quad \left(\because t_q = L_q/\lambda\right)$$

or, Expected time in the System = Expected time in queue + Time in Servic

$$\Rightarrow \boxed{t_s = t_q + 1/\mu}$$

→ Probability that the queue is non-empty.

$$P(n>1) = 1 - P_0 - P_1 = 1 - \left(1 - \frac{\lambda}{\mu}\right) - \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) = \left(\frac{\lambda}{\mu}\right)^2$$

→ Average length of non-empty queue (length of queue that is formed from time to time).

For a non-empty queue, the number of units in the system should be at least 2 (one in service and the others in the queue).

$$\therefore \text{Avg. length of non-empty queue} = \frac{\text{Avg. length of queue}}{\text{Probability of non-empty queue}} = \frac{\mu}{\mu - \lambda}$$

Relationships among Operating Characteristics of an M/M/1 queueing system

$$L_s = \frac{\lambda}{\mu - \lambda}, \quad L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}, \quad t_s = \frac{1}{\mu - \lambda}, \quad t_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

- Expected no. of customers in the system is equal to the expected no. of customers in the queue plus a customer currently in service, i.e. $L_s = L_q + \frac{\lambda}{\mu}$
- Expected waiting time of a customer in the system is equal to the expected waiting time in the queue plus the expected service time of a customer in service i.e., $t_s = t_q + \frac{1}{\mu}$
- Expected no. of customers in the system is equal to the average number of arrivals per unit of time multiplied by the avg. time spent by the customer in the system. i.e., $L_s = \lambda \cdot t_s$
- Expected no. of customers in the queue is equal to the avg no. of arrivals per unit time multiplied by the avg. time spent by a customer in the queue, i.e. $L_q = \lambda \cdot t_q$

✱ Relations between Average Queue Length and Average Waiting Time are known as Little's formulae.

Relation between 't_s' & 't_q' :-

$$t_s = \frac{L_s}{\lambda} = \frac{L_q+\rho}{\lambda} = \frac{L_q}{\lambda} + \frac{\rho}{\lambda} = \frac{L_q}{\lambda} + \frac{\lambda/\mu}{\lambda}$$

$$= \frac{L_q}{\lambda} + \frac{\lambda}{\mu \cdot \lambda} = \frac{L_q}{\lambda} + \frac{1}{\mu} = t_q + \frac{1}{\mu}$$

$$\therefore \boxed{t_s = t_q + \frac{1}{\mu}}$$

Or, Expected time spent in the system = espected time spent in queue + service time

Service rate (μ) = '6' customers/time (say)

then,
Service time (t_s) = $\frac{1}{\mu} = \frac{1}{6}$

Main Formuli are:

$\boxed{L_s = \frac{\rho}{1-\rho}}$ $\boxed{L_q = L_s - \rho}$ $\boxed{L_q = L_s - \rho = \frac{\rho^2}{1-\rho}}$

$\boxed{t_s = \frac{L_s}{\lambda}}$ $\boxed{t_q = \frac{L_q}{\lambda}}$ $\boxed{L_q = \lambda t_q}$

$\boxed{t_s = t_q + \frac{1}{\mu}}$ $\boxed{L_s = L_q + \rho}$

→ Relation between Queue length and time spent in the queue is known as 'Little's Formula'.

i.e $\boxed{L = \lambda W}$

where, L = Average number of items in a system or Length of queue

λ = Average arival rate

W = Average wait time in the system for an item

$W_s = t_s$ = Avg. waiting time/espected time spent in the system

$W_q = t_q$ = Avg. waiting time/espected time spent in the queue.

Problem-1

In a super market, the average rate of arrival of customer is 10 in every 30 minutes following Poison's process. The average time taken by a cashier to list and calculate the customer's purchase is 2.5 minutes following exponential distribution, Determine (i) Traffic intensity (ii) Length of the system (iii) Time spent in the queue and time spent by a customer in the system (iv) Probability of queue length that exceeds 6.

Solⁿ: Mean arrival rate, $\lambda = \frac{10}{30}$ customers/min

$= \frac{1}{3}$ customers/min $= \frac{1}{3}\ min^{-1}$

Service time $= 2.5$ min $= \frac{1}{\mu}$

Service rate $= \mu = \frac{1}{2.5}$ customer/min

(i) Traffic intensity (ρ)

$$\rho = \frac{\lambda}{\mu} = \frac{1/3}{1/2.5} = 0.83 \quad \text{(Ans)}$$

(ii) Length of the system, Ls

$$L_s = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83} = 4.88 \quad \text{(Ans)}$$

(iii) (a) Time spent in the queue, tq

$$t_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/3}{\frac{1}{2.5}\left(\frac{1}{2.5}-\frac{1}{3}\right)} = 12.5 \quad \text{(Ans)}$$

(b) Expected time spent by a customer in the system, ts

$$t_s = \frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{2.5}-\frac{1}{3}} = 15$$

(iv) Probability of queue length that exceeds 6

$$\rho^N = (0.83)^6 = 0.32$$

Problem- 2

People arrive at a theatre ticket counter in a poisson's distribution arrival rate of 25/hrs. Service time by the ticket issuing person is constant and equals to 2 mins. Calculate (i) Utility factor (ii) expected number of customer in the queue (iii) expected time spent in the system & queue (iv) Probability that there will no people/customer arrive at the ticket counter (v) probability of 10 peoples in the system.

Sol Given data

Arrival rate = 25/hours = 25/60 mins

Mean arrival rate, $\lambda = \frac{25}{60} = \frac{5}{12}$ customer/min

Service time, $\frac{1}{\mu} = 2$ min

Service rate, $\mu = \frac{1}{2}$ customer/min

(i) Utilization factor/Traffic intensity:-

$$\rho = \frac{\lambda}{\mu} = \frac{5/12}{1/2} = 0.83 \text{ or } 83\% \quad \text{(Ans)}$$

(ii) Expected number of customer in the queue

$$L_q = L_s - \rho = \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu} = \frac{5/12}{1/2 - 5/12} - \frac{5/12}{1/2} = 4.16 \quad \text{(Ans)}$$

(iii) (a) Expected time spent in the system

$$t_s = \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{2}-\frac{5}{12}} = 12 \quad \text{(Ans)}$$

(b) Expected time spent in the queue

$$t_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{5/12}{\frac{1}{2}\left(\frac{1}{2}-\frac{5}{12}\right)} = 10 \quad \text{(Ans)}$$

(iv) Probability of no people/customer arrive at the ticket counter

$$P_0 = 1-\rho = 1-0.83 = 0.17 \text{ or } 17\% \quad \text{(Ans)}$$

(V) Probability of 10 peoples in the system

$$P_n = \rho^n (1-\rho)$$
$$= \rho^{10}(1-0.83)$$
$$= (0.83)^{10}(1-0.83)$$
$$= 0.026 \text{ or } 2.6\% \quad \text{(Ans)}$$

Problem-3

In a railway marshalling yard goods trains arrive according to Poison's distribution with a mean arrival rate 30 train/day. Assuming that the service time is approximately exponential with an average time of service of 36 minute for each train. Calculate the following;

(i) Utilization factor

(ii) Expected number of train in the system and mean (avg.) queue size.

(iii) The probability that the queue size exceeds 10.

(iv) If the arrival of the goods trains increases to an average of 33/day what will be changes in (i) & (ii)

Sol.^n Assuming the given queuing model as queuing model with single server, infinity capacity.

Given data

Mean arrival rate, $\lambda = 30/\text{day} = \frac{30}{60\times24} = \frac{1}{48}$

or $\lambda = \frac{1}{48}$ trains/min

Mean service rate, $\mu = \frac{1}{30}$ trains/min

(i) Utilization Factor (ρ)

$$\rho = \frac{\lambda}{\mu} = \frac{1/48}{1/36} = \frac{3}{4} \text{ or } 0.75 \quad \text{(Ans)}$$

(ii) Expected number of goods trains in the system

$$L_s = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3 \quad \underline{\text{Ans}}$$

b. Mean queue size

$$L_q = L_s - \rho = 3 - 0.75 = 2.25 \quad \underline{\text{Ans}}$$

(iii) Probability that queue size exceeds 10

$$\rho^N = (0.75)^{10} = 0.056 \text{ or } 5.6\%$$

(iv) New mean arrival, $\lambda' = 33$ trains/day

$$\lambda' = \frac{33}{60 \times 24} = 0.022 \text{ trains/min}$$

New value of utilization factor, ρ'

$$\rho' = \frac{\lambda'}{\mu} = \frac{0.022}{1/36} = 0.792 \text{ or } 79.2\%$$

$\underline{\text{Ans}}$

New value of Length of the system, L_s'

$$L_s' = \frac{\rho'}{1-\rho'} = \frac{0.79}{1-0.79} = 3.76 \quad \underline{\text{Ans}}$$

New mean queue size, L_q'

$$L_q' = L_s' - \rho' = 3.76 - 0.79 = 2.97 \quad \underline{\text{Ans}}$$

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Problem - 4

A person repairing radios finds that the time spent on the radio sets has exponential distribution with mean service time of 20min for each radio. If the radios are repair in the order in which they come in and their arrival is approximately poisson's distribution with an average rate of 15 for 8hours a day. What is the reference expected idle time each day? How many jobs are ahead (Ls) of the average set just brought in?

Sol^{n} Assuming the given queuing model as single service and infinity capacity.

Given data

Arrival rate = 15 radio/8hrs

Mean arrival rate, $\lambda = \frac{15}{8}$ radios/hrs

Service rate = 20min/radios

20 minutes $\longrightarrow$ 1 radio

(20×3) min $\longrightarrow$ (1×3) radios

60min or 1hrs $\longrightarrow$ 3 radios

Mean Service rate, μ = 3 radios/hr.

In 1 hour the repair man remains busy

$$\rho = \frac{\lambda}{\mu} = \frac{15/8}{3} = \frac{5}{3} = 0.625 \text{ or } 62.5\%$$

Number of hours during which the repair man remain busy in a 8 hour working day,

$$= 8 \times 0.625 = 5\text{hrs}$$

Repair man's idle time = 8 − 5 = 3 hrs. (Ans)

$$Ls = \frac{\rho}{1-\rho} = \frac{0.625}{1-0.625} = 1.66$$

Hence, 1.66 jobs are ahead of the avg. set just brought in.

MODEL-II (M/M/1):(N/FCFS):-

[Single server, Finite Queuing capacity]

→ Traffic intensity, $\boxed{\rho = \frac{\lambda}{\mu}}$

→ Probability of no units or customers in the system, P_0

$$\boxed{P_0 = \frac{1-\rho}{1-\rho^{N+1}}} \quad (= \text{idle time})$$

→ Probability of 'n' units or customers in the system, P_n

$$\boxed{P_n = \left(\frac{1-\rho}{1-\rho^{N+1}}\right) \cdot \rho^n}$$

$\begin{cases} n = \text{Number of customer} \\ N = \text{Capacity of the system} \end{cases}$

→ Espected length of the system, Ls

$$\boxed{Ls = P_0 \cdot \sum_{n=0}^{N} n \cdot \rho^n}$$

→ Espected length of the queue, Lq

$$\boxed{Lq = Ls - \rho}$$

→ Mean time spent in the queue, t_q

$$\boxed{tq = \frac{Lq}{\lambda}}$$

→ Mean time spent in the system, t_s

$$\boxed{ts = tq + \frac{1}{\mu} = \frac{Ls}{\lambda}}$$

Problem-1

A car park can accommodates 5 cars. The arrival of cars is Poison at a mean arrival rate of 10 cars per hours. The length of the time that each cars spents in the car park is exponential distribution with mean of 5 hrs. How many cars are in the car park on an average?

sol^n Given data

Here queuing model is single server and finite capacity of the system

Mean arrival rate, $\lambda = 10$ cars/hr.

service rate $= 5$ hrs

Mean service rate, $\mu = \frac{1}{5}$ cars/hr.

Traffic intensity, ρ

$$\rho = \frac{\lambda}{\mu} = \frac{10}{1/5} = 50$$

Probability of no car in the system, P_0

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-50}{1 \cdot 50^{5+1}} = 3.136 \times 10^{-9}$$

$$\approx 0$$

Number of car in the car park

$$L_s = P_0 \cdot \sum_{n=0}^{N} n \cdot \rho^n = 0 \cdot \sum_{n=0}^{5} n \cdot 50^n$$

$$= 0$$

$\therefore$ Zero (0) car is in the car park on an average.

Problem-2

In railway marshalling yard, goods trains arrive at the rate of 30 trains/day. Assume that the interval time follows an exponential distribution and the service time is also be assumed as exponential with mean of 36 minutes. Calculate following

(i) Probability that of the yard is empty

(ii) The average queue length assuming that the line capacity of the yard is 9 trains.

Sol^n: Given data

arrival rate = 30 trains/day

Mean arrival rate, $\lambda = \frac{30}{24 \times 60} = \frac{1}{48}$ trains/min

capacity of yard is $N = 9$ trains

service rate = 36 minutes

Mean service rate, $\mu = \frac{1}{36}$ trains/min

Traffic intensity, $\rho = \frac{\lambda}{\mu} = \frac{1/48}{1/36} = 0.75$ or 75%

(i) Probability that the yard is empty, P_0

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^{9+1}} = 0.265 \text{ or } 26.5\%$$

(Ans)

(ii) Average queue length, L_s

$$L_s = P_0 \cdot \sum_{n=0}^{N} n \cdot \rho = 0.265 \times \sum_{n=0}^{9} n \cdot (0.75)^n$$

$$= 0.265 \times 9.07$$

$$= 2.404 \quad \text{(Ans)}$$