ONLINE CLASS #01 Subject: MECHANICAL VIBRATIONS

DATE - 23-07-2020 : Chapter-1: Introduction to MV

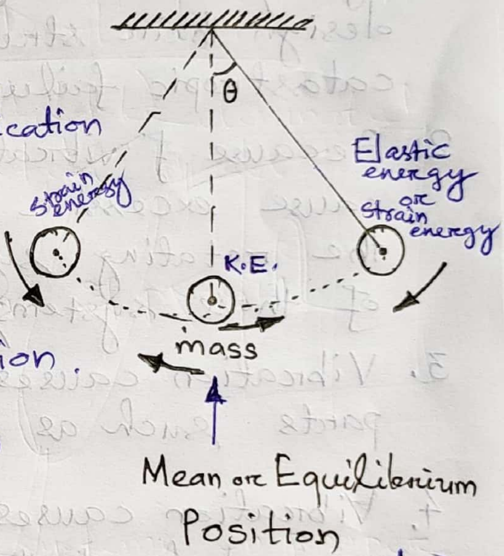
Vibration: It is a periodic back-and-forth motion of the particles of an elastic body or medium, commonly resulting when almost any physical system is displaced from its equilibrium or mean position.

⇨ When such vibrations occurs in mechanical components or machineries, it is called as 'Mechanical Vibration'.

How Vibration Initiates?

- All bodies having mass and elasticity are capable of vibration.
- The mass is inherent of the body and elasticity causes relative motion among its particles.

When body particles are displaced from their respective mean or equilibrium position by the application of external force, then the internal forces (in the form of elastic energy) try to bring the body to its original position.



⇨ When the mass/body displaces from its original rest position (due to external force), elastic energy or strain energy is stored in the body at its disturbed/displaced position.

⇨ but the internal forces (generated by the particles) try to ~~the~~ bring the body to original position; thereby at equilibrium position, all strain energy is converted into Kinetic energy (K.E)

⇨ ~~Be~~ As there is K.E. with the body, the body will not stop at the mean position; rather it will move towards opposite dirn. so that all the K.E. will be converted into strain energy at displaced position.

In this way the body attains in continous motion (or oscillation) which creates vibration.

Main reasons of Vibrations :

1. Elastic nature of the system
2. Unbalanced centrifugal force in the system (this arises due to non-uniform material distribution in a rotating machine element.)
3. External excitation applied to the system
4. Wind may cause vibrations of certain system such as elastric transmission lines, telephone lines, etc.

Why should we study 'Mechanical Vibration' in Engineering?

We discuss this aspect in following two ways:

Ⓐ Undesirable areas of Vibration

1. The structure designed to support the high speed engines/ machines (such as turbines, Lathe, etc.) unless design their structure suitably, leads to catastropic failure.
2. Because of vibrations, faulty design and manufacturing cause excessive & unpleasant stresses in the rotating system which leads to failure of those systems/ components.
3. Vibration causes rapid weare of the machine parts such as bearing and gears.
4. Vibration causes Loosening of parts from the main body of the machine which may lead to accidents.
5. Because of excessive deformation due to vibrations, the wheel of the locomotive can leave the track which results ~~on~~ in accident and heavy loss of life and the system.
6. Building structure, bridges, fail because of vibration.

Thus, the study of vibration is essential for an design engineer to minimize the vibration effects over mechanical components by design them suitably.

Scanned with CamScanner

(B) Desirable area of Vibration

Vibration can be used for fruitful purposes in day to day life even in mechanical fields.

- → Musical instruments
- → Earthquake's for geological research.
- → It is useful for propagation of sound.
- → In body vibrator tool.
- → Construction work (Mixture m/c, vibrator during concrete work)
- → Testing equipments like vibratory conveyors, hoppers, sievers and compactors.
- → Machining processes such as in USM.

Methods to minimise undesirable vibrations

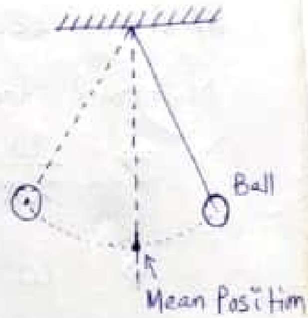
The undesirable vibrations should be eliminated or reduced upto certain extent by several methods.

1. Removing external excitation if possible
2. Using shock absorbers
3. Using dynamic absorbers
4. Resting and mounting the systems on proper vibration isolators (such as springs, etc.).

Terminology Of Vibration

Or, Terms Used In Vibration

Simple Harmonic Motion (SHM) : The motion of a body to and fro about a fixed point is called simple harmonic motion. The motion is periodic and its acceleration is always directed towards the mean position and is proportional to its distance from mean position.



(SHM)

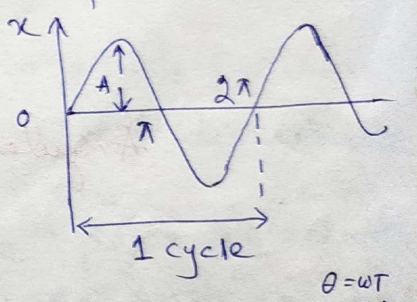
Let a body having SHM is represented by the equation, $x = A \sin \omega t$

Velocity, $\dot{x} = A\omega . \cos \omega t$

Acceleration, $\ddot{x} = -A\omega^2 . \sin \omega t$

where A = amplitude of vibration

* Time Period (T) : It is the time taken to complete one cycle of motion/vibration. ~~It equals~~



$$\omega \cdot T = 2\pi$$

$$\Rightarrow \boxed{T = \frac{2\pi}{\omega}}, \text{ sec}, \quad \omega = \text{rad/sec}.$$

Frequency of Oscillation (f) :— Frequency is the number of cycles of motions completed in one second.

$$\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi}}, \text{ cycle/sec or hertz (Hz)}$$

Problem-1 : A harmonic motion is expressed as $x = 1.25 \sin\left(15\pi t - \frac{\pi}{3}\right)$ where x is measured in cm, t in seconds and the angle in radians. For this motion determine

(i) Frequency (ii) Time period (iii) Maximum Displacement (m) (iv) Max^m velocity (v) Max^m Acceleration (m/s^2).

Solution : We have, $x = 1.25 \sin\left(15\pi t - \frac{\pi}{3}\right)$

Here, $\omega = 15\pi$, $A = 1.25$, $\phi = \frac{\pi}{3}$

$$\boxed{\begin{array}{l} x = A \sin(\omega t - \phi) \\ \text{or } x = X \cdot \sin(\omega t - \phi) \end{array}}$$

(i) Frequency, $f = \frac{\omega}{2\pi} = \frac{15\pi}{2\pi} = 7.5$ cycle/sec.

(ii) Time period, $T = \frac{1}{f} = \frac{1}{7.5} = 0.133$ sec.

(iii) Max^m Displacement, $A = 1.25$ cm $= 0.0125$ m

(iv) Max^m Velocity, $= A\omega$
$= 0.0125 \times 15\pi = 0.589$ m/sec.

(v) Max^m Acceleration $= A \cdot \omega^2$
$= 0.0125 \times (15\pi)^2 = 27.8$ m/sec. Ans

Problem-2 :- An instrument has a natural frequency of 10 Hz. It can stand a maximum acceleration of 10 m/s^2. Find maximum amplitude of displacement.

Solution :- We have that $x = A \cdot \sin\omega t$

$$\ddot{x} = -A\omega^2 \sin\omega t$$

Maximum acceleration $= \omega^2 A = 10 \ m/s^2$ ——— (1)

Natural frequency $= 10$ Hz ——— (2)

$$\Rightarrow \frac{\omega}{2\pi} = 10$$

$$\Rightarrow \omega = 2\pi \times 10 = 20\pi \ rad/s$$

$\therefore$ Max^m amplitude of displacement, $A = \frac{10}{\omega^2}$ from equ^n(1)

$$= \frac{10}{(20\pi)^2} = 2.53 \ mm.$$

Ans

Problem-3 : A harmonic motion is given by $x(t) = 10 \sin\left(30t - \frac{\pi}{3}\right)$ mm where it is in seconds and phase angle in radians. Find (i) frequency and the period of motion, (ii) the max^m displacement, velocity and acceleration.

Solution: We have $x(t) = 10 \sin\left(30t - \frac{\pi}{3}\right)$

Here, $A = 10$ mm, $\omega = 30$ rad/sec, $\phi = \frac{\pi}{3}$.

(i) frequency $(f) = \frac{\omega}{2\pi} = \frac{30}{2\pi} =$

Period of motion, $T = \frac{1}{f} =$

(ii) Max^m Displacement, $A = 10$ mm

Max^m velocity $= A \cdot \omega = 10 \times 30 = 300$ mm/sec.

Max^m Acceleration $= A \cdot \omega^2 = 10 \times (30)^2 = 9000$ mm/sec^2.

Ans

* Time period, $T = \frac{\text{Wavelength}}{\text{velocity}}$

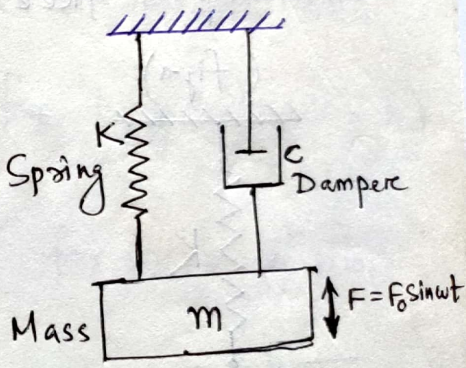
Angular frequency, $\omega = \frac{2\pi}{T}$

* Elements of Vibratory System

The main elements of a vibratory system are

1- the mass
2- the spring
3- the damper
4 the excitation element ← (Active)

(Passive)



(fig: A typical vibratory sys)

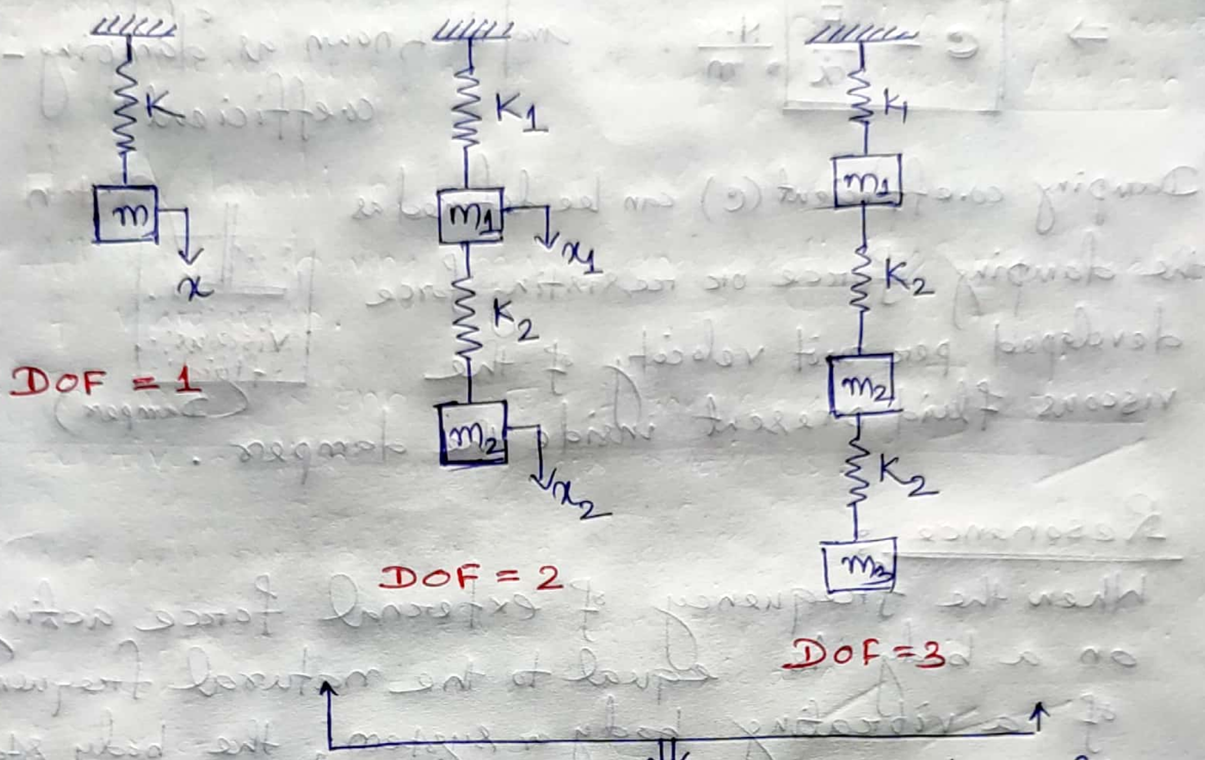
→ The first three elements are called passive elements which are used to define a system.

Mass and spring are used to store energy whereas damper is for dissipation of energy.

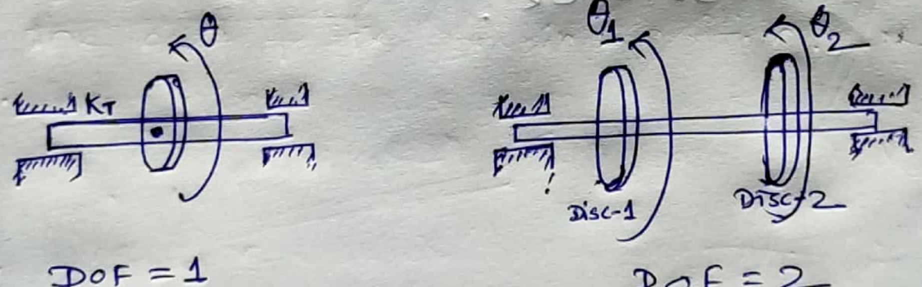
→ The excitation is called an active element.

Degree of Freedom (DOF) :

The minimum no. of coordinates required to specify the motion of a vibrating system at any instant is called as 'degree of freedom (DOF)'.



Multi-degree of freedom mechanical vibration systems

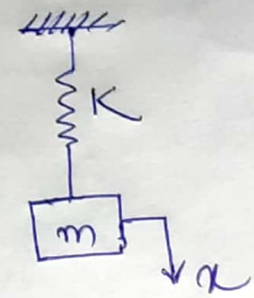


Stiffness of Spring (K) or Spring Constant :

$$F \propto x$$

$\Rightarrow$ $F = K \cdot x$, K = proportionality constant and is known as stiffness of spring or spring constant.

$\Rightarrow$ $\boxed{K = \frac{F}{x}}$, N/m

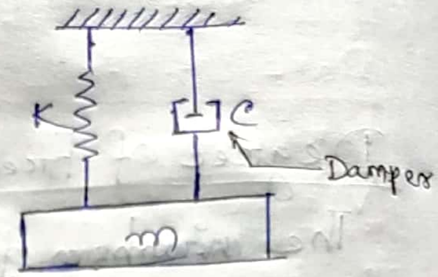


i.e, It is defined as force required to produce unit displacement is called stiffness of spring.

Damping :

Let F = force applied on damper

V or $\dot{x}$ = velocity of viscous fluid

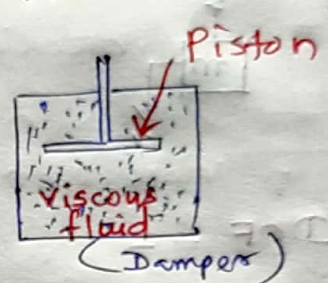


Then, $F \propto \dot{x}$

$\Rightarrow F = c\dot{x}$, c = proportionality constant and is known as 'damping-coefficient'.

$\Rightarrow \boxed{c = \frac{F}{\dot{x}}}$, $\frac{N\text{-}s}{m}$

Damping co-efficient (c) can be defined as the damping force or resisting force developed per unit velocity of the viscous fluid present inside the damper.



Resonance :

When the frequency of external force acting on a body is equal to the natural frequency of a vibrating body or system, the body starts vibrating with large amplitude. Such a state is known as resonance.

23.3. Types of Vibratory Motion

The following types of vibratory motion are important from the subject point of view :

1. ***Free or natural vibrations.*** When no external force acts on the body, after giving it an initial displacement, then the body is said to be under ***free or natural vibrations.*** The frequency of the free vibrations is called ***free or natural frequency.***

2. ***Forced vibrations.*** When the body vibrates under the influence of external force, then the body is said to be under ***forced vibrations.*** The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

Note : When the frequency of the external force is same as that of the natural vibrations, resonance takes place.

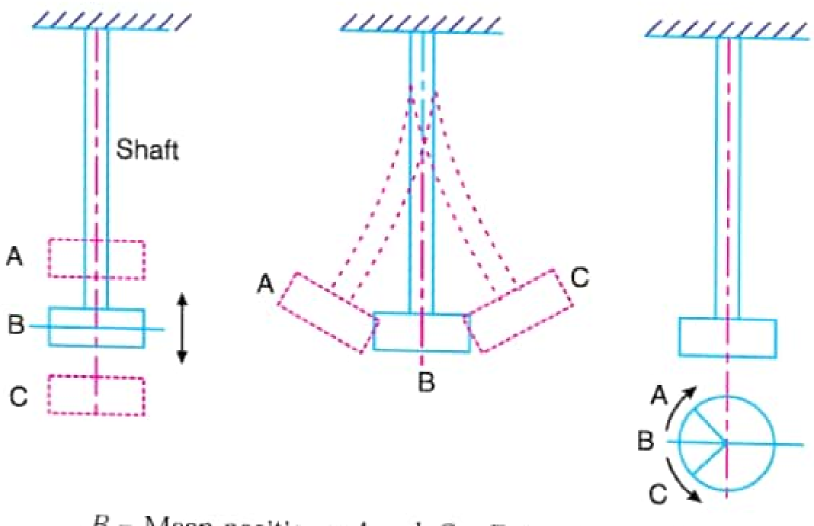
3. ***Damped vibrations.*** When there is a reduction in amplitude over every cycle of vibration, the motion is said to be ***damped vibration.*** This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

23.4. Types of Free Vibrations

The following three types of free vibrations are important from the subject point of view :

1. Longitudinal vibrations, **2.** Transverse vibrations, and **3.** Torsional vibrations.

Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig. 23.1. This system may execute one of the three above mentioned types of vibrations.



B = Mean position ; *A* and *C* = Extreme positions.

(*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations.

Fig. 23.1. Types of free vibrations.

1. ***Longitudinal vibrations.*** When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig. 23.1 (*a*), then the vibrations are known as ***longitudinal vibrations.*** In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.

2. ***Transverse vibrations.*** When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig. 23.1 (*b*), then the vibrations are known as ***transverse vibrations.*** In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft.

3. *Torsional vibrations**. When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig. 23.1 (*c*), then the vibrations are known as ***torsional vibrations.*** In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

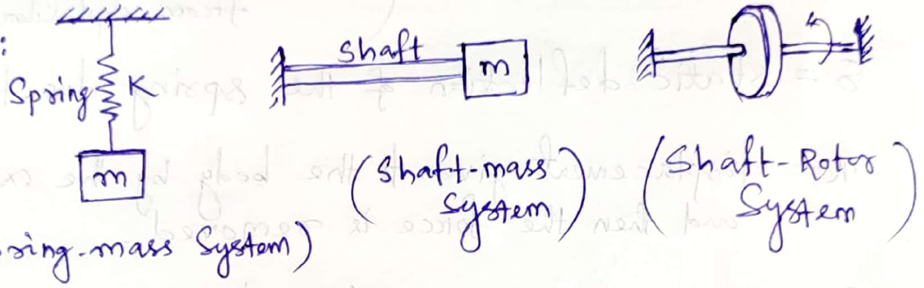
Note : If the limit of proportionality (*i.e.* stress proportional to strain) is not exceeded in the three types of vibrations, then the restoring force in longitudinal and transverse vibrations or the restoring couple in torsional vibrations which is exerted on the disc by the shaft (due to the stiffness of the shaft) is directly proportional to the displacement of the disc from its equilibrium or mean position. Hence it follows that the acceleration towards the equilibrium position is directly proportional to the displacement from that position and the vibration is, therefore, simple harmonic.

Chapter-2 : Undamped Free Vibrations

- When an elastic system vibrates because of inherent forces and no external force acts on the body, after giving it an initial displacement, it is called free vibration.
- If there is no damping element in the system and thereby no decay or reduction of vibration over every cycle due to frictional resistance, the vibration is known as undamped vibration.

⇨ The combination of the above two types of vibrations is called as 'undamped free vibration'.

Example:



Natural Frequency of Free Longitudinal Vibrations

The natural frequency (ω_n or f_n) of the undamped free longitudinal vibrations can be determined by the following three methods:

(ω_n → rad/s, f_n → Hz)

1. Equilibrium Method (or Newton's Method)
2. Energy Method (or Robert Mayer's Method)
3. Rayleigh Method (or Max^m Energy Method)

① Equilibrium Method :

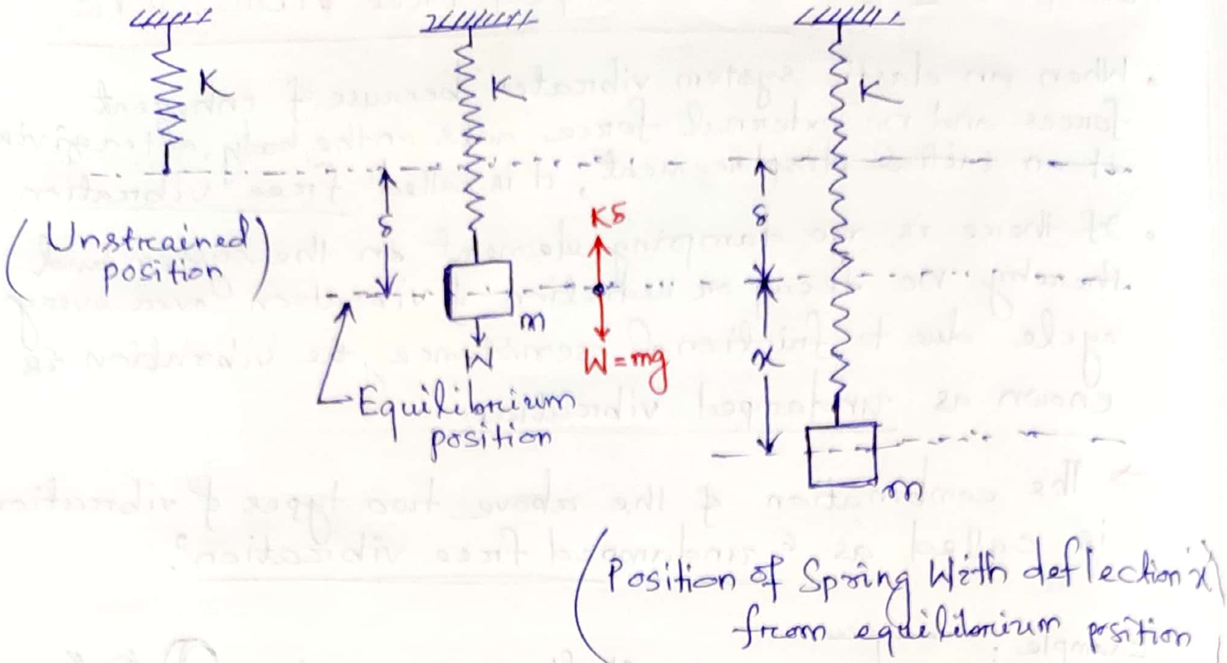
Consider a spring-mass system, initially in an unstrained (or undisturbed) position.

Assumption: Spring has negligible mass of its own.

Let K = spring constant or stiffness of the spring, N/m

m = mass of the body suspended from the spring, kg

W = weight of the body = mg.



δ = static deflection of the spring due to weight 'W'

x = displacement given to the body by the external force and then the force is removed.

Since the mass is now displaced from its equilibrium position by a distance 'x' and is then released,

$$\text{Restoring force} = W - K(\delta + x)$$

$$= \cancel{K\delta} - \cancel{K\delta} - Kx \quad \left(\because K = \frac{W}{\delta} \Rightarrow W = K\delta\right)$$

$$= -K.x \; (\uparrow) \qquad \text{acting in upward dir}^n$$

and

$$\text{Accelerating force} = \text{Mass} \times \text{Acceleration}$$

$$= m.\ddot{x} \; (\downarrow) \qquad \text{acting downward}$$

Equating above two equations,

$$m\left(\frac{d^2x}{dt^2}\right) = -K.x$$

$$\Rightarrow m\left(\frac{d^2x}{dt^2}\right) + Kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \left(\frac{K}{m}\right)x = 0$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega_n^2 x = 0}$$

Or

$$\boxed{\ddot{x} + \omega_n^2 x = 0}$$

Equation of motion of Undamped Free Vibrations where, $\omega_n^2 = \frac{K}{m}$

$$\Rightarrow \boxed{\omega_n = \sqrt{\frac{K}{m}}}\ \text{, rad/sec}$$

Or, $$\boxed{f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}}\ \text{, hertz}$$

Natural frequency of Undamped free-Longitudinal Vibrations.

$$* \boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}}$$

$\left(\because W = mg = K \cdot \delta \Rightarrow \frac{K}{m} = \frac{g}{\delta} \right)$

(In terms of static deflection of the spring or other elastic constraint which carries a mass at its one end)

② Energy Method :

We know that

Kinetic energy (K.E.) is due to motion of the body and Potential energy (P.E.) is due position of the body with respect to a certain datum/reference position.

[In case of vibrations, the datum position is the mean or equilibrium position at which the potential energy of the body or system is ZERO]

⇒ In free vibrations, no energy is transferred to the system or from the system, then according to conservation of energy,

$$(K.E + P.E.) = \text{constant at all the time.}$$

$$\therefore \frac{d}{dt}(K.E + P.E.) = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2} \times m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} K x^2 \right] = 0$$

$\left(\because P.E. = \text{Mean force} \times \text{Displacement} = \left(\frac{0 + K \cdot x}{2}\right) \times x = \frac{1}{2} K x^2 \right.$

$$\Rightarrow \frac{1}{\cancel{2}} \times m \times \cancel{2}\left(\frac{dx}{dt}\right) \times \frac{d^2x}{dt^2} + \frac{1}{\cancel{2}} K \times \cancel{2}x \times \frac{dx}{dt} = 0$$

$$\Rightarrow \left[m\left(\frac{d^2x}{dt^2}\right) + K.x\right] \cdot \frac{dx}{dt} = 0$$

$$\Rightarrow m\left(\frac{d^2x}{dt^2}\right) + K.x = 0$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \left(\frac{K}{m}\right)x = 0}$$

Or $$\boxed{\frac{d^2x}{dt^2} + \omega_n^2 x = 0}$$

Or $$\boxed{\ddot{x} + \omega_n^2 x = 0}$$

Differential Equ^n of motion of Undamped free-longitudinal vibrations.

where natural frequency $= \omega_n = \sqrt{\frac{K}{m}}$, rad/s

Or $\boxed{f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}}$, rad/s

③ Rayleigh's Method :

In this method,

$$\boxed{\text{Maximum K.E. at the mean position} = \text{Max}^m \text{ P.E. at the extreme position.}}$$

Assuming the motion executed by the vibration to be simple harmonic motion (SHM), then

$$x = X \sin \omega_n t \quad \text{——— ①}$$

where, x = displacement of the body from the mean position after time 't'.

X = Maximum displacement from mean position to the extreme position (sometimes, we take 'A')

Velocity at the mean position, $\dot{x} = \frac{dx}{dt} = \frac{d}{dt}(X \sin \omega_n t)$

$$\Rightarrow \dot{x} = \omega_n X \cdot \cos \omega_n t$$

$\therefore$ Max^m kinetic energy at mean position $= \frac{1}{2} m (\dot{x}_{max})^2$

$$\Rightarrow (K.E_{max})_{\text{Mean position}} = \frac{1}{2} m (\omega_n X)^2 \quad \text{——— ②}$$

[$\because$ $\dot{x} = \omega X \cdot \cos \omega t$

At mean position, phase angle $= 0$, So, velocity ($\dot{x}$) becomes max^m.

Thus, $\dot{x}_{max} = \omega_n X$ as $\cos 0^\circ = 1$]

Max^m Potential energy at the extreme position

$= $ Mean force $\times$ Max^m displacement

$= \left(\frac{0 + K.X}{2}\right) \times X$

$$(P.E_{max})_{\text{Extreme position}} = \frac{1}{2} K X^2 \quad \text{——— ③}$$

$\because K = \frac{\text{force}}{\delta}$

$\Rightarrow$ Force $= K \cdot \delta$

$\Rightarrow$ Max^m force $= K \cdot \delta_{max}$
$= K \cdot X$

Scanned with CamScanner

Equating equations ② & ③,

$$\frac{1}{2} m \omega_n^2 X^2 = \frac{1}{2} K \cdot X^2$$

$$\Rightarrow \omega_n^2 = \frac{K}{m}$$

$$\Rightarrow \boxed{\omega_n = \sqrt{\frac{K}{m}}} \text{, rad/sec}$$

Or, $\boxed{f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}}$, Hz

Natural frequency of Damped free-Longitudinal Vibrations

Date: 04.08.2020 ⸰ ONLINE CLASS #06 ⸰ Subj: MV

Chapter-02 : Topic : Natural frequency of Free Transverse Vib.

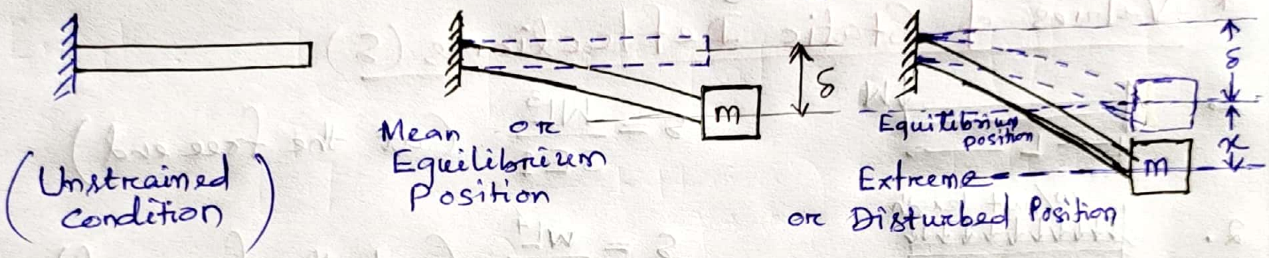
Consider a shaft of negligible mass, whose one end is fixed and the other end carries a body of mass 'm'.

Let K = stiffness of shaft

δ = Static deflection due to weight of the body

x = displacement of body from mean or equilibrium position (due to external disturbance) after time 't'

W = weight of the mass $= m.g$



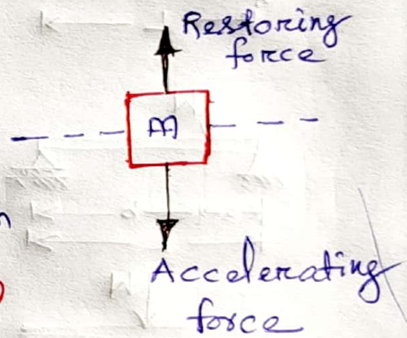
Restoring force $= W - K(\delta + x)$

$= K.\delta - K.\delta - K.x$

$= -K.x$ —— ①

Accelerating force = mass × acceleration

$= m \times \frac{d^2x}{dt^2}$ —— ②



(fig: Free Body Diagram of the mass attached)

Equating equ.ⁿ ① & ②, the equ.ⁿ of motion becomes,

$$m\left(\frac{d^2x}{dt^2}\right) = -K.x$$

$$\Rightarrow m\left(\frac{d^2x}{dt^2}\right) + K.x = 0$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \left(\frac{K}{m}\right)x = 0}$$

or $$\boxed{\frac{d^2x}{dt^2} + \omega_n^2 . x = 0}$$

or $$\boxed{\ddot{x} + \omega_n^2 . x = 0}$$

Differential Equation of motion of Undamped free Transverse Vibrations.

Natural frequency of the free transverse vibrations,

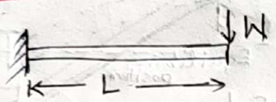
$$\omega_n^2 = \frac{K}{m} \Rightarrow \boxed{\omega_n = \sqrt{\frac{K}{m}}},\ \text{rad/s}$$

Or $\boxed{f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}}$, Hz

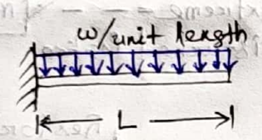
Also, $\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}}$ $\left(\because W = mg = K\cdot\delta \Rightarrow \frac{K}{m} = \frac{g}{\delta}\right)$

where, δ = static deflection due to weight force of the body attached to the shaft or beam.

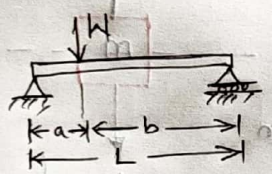
* Values of Static Deflections (δ)

1. 

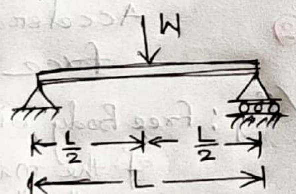
$\delta = \frac{WL^3}{3EI}$ (at the free end)

2. 

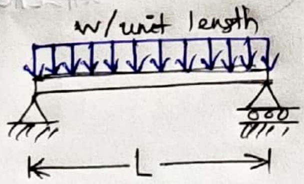
$\delta = \frac{wL^4}{8EI}$ (at the free end)



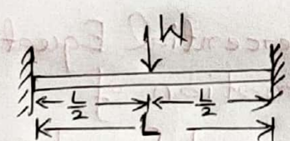
$\delta = \frac{Wa^2b^2}{3EIL}$ (at the point of action of Load)



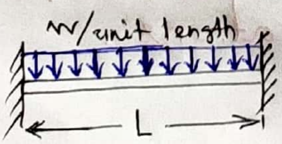
$\delta = \frac{WL^3}{48EI}$ (at the midpoint)



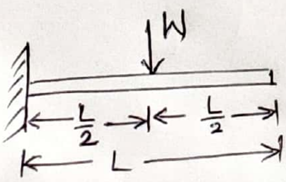
$\delta = \frac{5wL^4}{384EI}$ (at the centre)



$\delta = \frac{WL^3}{192EI}$ (at the mid-point)



$\delta = \frac{wL^4}{384EI}$ (at the centre)

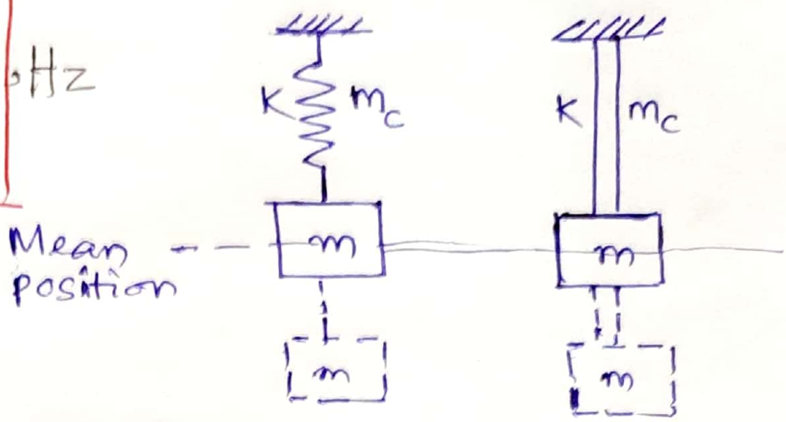


$\delta = \frac{WL^3}{192EI}$ (at the centre)

* Effect Of Inertia of The Constraint

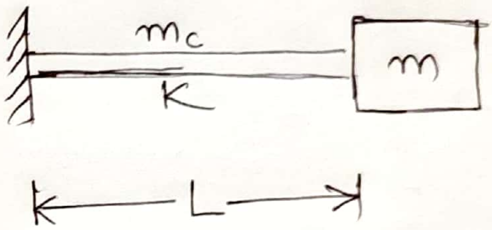
(A) ~~Longitudinal~~ free Longitudinal Vibrations

$$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{K}{\left(m + \frac{m_c}{3}\right)}}}\ , \text{Hz}$$

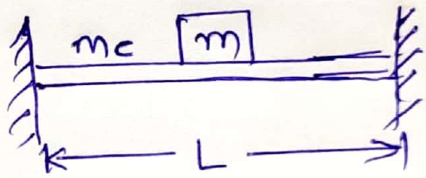


(B) Free Transverse Vibrations

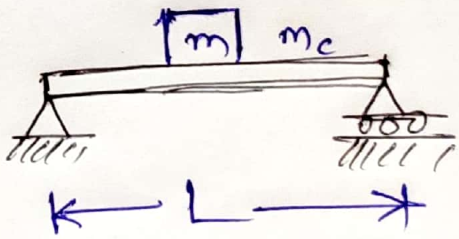
$$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{K}{\left(m + \frac{33}{140} m_c\right)}}}\ , \text{Hz}$$



$$* \quad f_n = \frac{1}{2\pi}\sqrt{\frac{K}{\left(m + \frac{13}{35} m_c\right)}}$$



$$* \quad f_n = \frac{1}{2\pi}\sqrt{\frac{K}{\left(m + \frac{17}{35} m_c\right)}}$$



Problem-1 : A cantilever shaft 50 mm diameter and 300 mm long has a disc of mass 100 kg at its free end. The Young's modulus for the shaft material is 200 GN/m^2. Determine

i) the natural frequency of longitudinal vibrations of the shaft.

ii) the ~~transverse~~ natural frequency of transverse vibrations of the shaft.

Solution: Given datas

$d = 50\ mm = 0.050\ m$; $E = 200\ GN/m^2 = 200 \times 10^9\ N/m^2$

$L = 300\ mm = 0.300\ m$; $m = 100\ kg$

$$A = \frac{\pi d^2}{4} = \frac{\pi (0.050)^2}{4} = 1.96 \times 10^{-3}\ m^2$$

$$I = \frac{\pi d^4}{64} = \frac{\pi}{64}(0.050)^4 = 0.3 \times 10^{-6}\ m^4$$

① **Natural frequency of longitudinal vibrations**

We know static deflection, $\delta = \frac{WL}{AE} = \frac{(100 \times 9.81) \times 0.300}{1.96 \times 10^{-3} \times 200 \times 10^9}$

$$= 0.751 \times 10^{-6}\ m$$

∴ Natural frequency of longitudinal vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.751 \times 10^{-6}}}$$

$$\Rightarrow \boxed{f_n = 575\ Hz} \quad \text{Ans}$$

(ii) **Natural frequency of transverse vibrations**

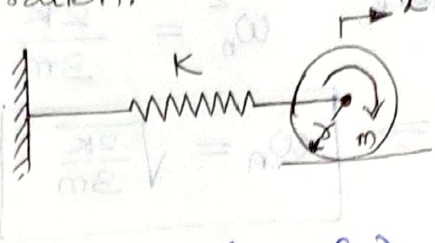
Static deflection of the shaft, $\delta = \frac{WL^3}{3EI} = \frac{100 \times 9.81 \times (0.3)^3}{3 \times 200 \times 10^9 \times 0.3 \times 10^{-6}}$

$$= 0.147 \times 10^{-3}\ m$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.147 \times 10^{-3}}} = 41\ Hz$$

Ans

Problem- : A circular cylinder of mass 4 kg and radius 15 cm is connected by a spring of stiffness 400 N/m as shown in the figure. It is free to roll on horizontal rough surface without slippling, determine the natural frequency of vibration.

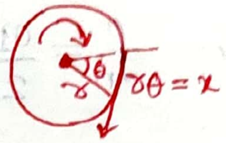


Solution: Given data

$m = 4\,kg$, $r = 15\,cm$

$K = 400\,N/m$

By energy method, the natural frequency (ω_n or f_n) of the given spring-roller/cylinder can be determined as follows:



$$K.E. + P.E. = \text{Constant}$$

$$\Rightarrow \left(\underset{\text{translatory motion}}{\text{K.E. due to}} + \underset{\text{rotational motion}}{\text{K.E. due to}}\right) + \text{P.E. of Spring} = C$$

$$\Rightarrow \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}Kx^2 = C$$

$$\Rightarrow \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}K(r\theta)^2 = C$$

$$\Rightarrow \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}\times\left(\frac{1}{2}mr^2\right)\dot{\theta}^2 + \frac{1}{2}Kr^2\theta^2 = C$$

$$\Rightarrow \frac{3}{4}mr^2\dot{\theta}^2 + \frac{1}{2}Kr^2\theta^2 = C$$

$(K.E.)_{Translation} = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$

$(K.E.)_{Rotational} = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\theta}^2$

$\because I_{\text{Circular Cylinder}} = \frac{1}{2}mr^2$

Differentiating both sides w.r.t time 't',

$$\frac{3}{4}mr^2\times 2\dot{\theta}\times\ddot{\theta} + \frac{1}{2}Kr^2\times 2\theta\times\dot{\theta} = 0$$

$$\Rightarrow \left[\frac{3}{2}mr^2\ddot{\theta} + Kr^2\theta\right]\dot{\theta} = 0$$

$$\Rightarrow \frac{3}{2}mr^2\ddot{\theta} + Kr^2\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{Kr^2}{\left(\frac{3}{2}mr^2\right)}\theta = 0$$

$$\Rightarrow \boxed{\ddot{\theta} + \left(\frac{2K}{3m}\right)\theta = 0} \qquad \text{——— (1)}$$

Compare equ^n (1) with general differential equ^n of motion of free longitudinal vibrations as

$$\boxed{\ddot{\theta} + \omega_n^2 \theta = 0}$$

(In terms of rotational co-ordinate).

Here, $\omega_n^2 = \frac{2K}{3m}$

$$\Rightarrow \boxed{\omega_n = \sqrt{\frac{2K}{3m}}}, \text{rad/s}$$

Or, $\omega_n = \sqrt{\frac{2 \times 4000}{3 \times 4}} = 25.82$ rad/sec

or, $f_n = \frac{\omega_n}{2\pi}$, Hz $= \frac{25.82}{2\pi} = 4.10$ Hz <u>Ans</u>

Example 2.32 *A 5 kg mass attached to the lower end of a spring, whose upper end is fixed, vibrates with a natural period of 0.45 sec. Determine the natural period when a 2.5 kg mass is attached to the mid point of the same spring with the upper and lower ends fixed.*

(P.U., 92)

Solution. Natural frequency can be

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.45} = 13.95 \text{ rad/sec}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 13.95$$

or $$(13.95)^2 = \frac{k}{m}$$

$$k = m\,(13.95)^2 = 5(13.95)^2 = 973 \text{ N/m}$$

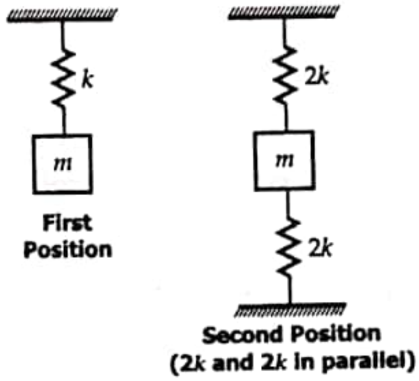


Fig. 2.36A

when the spring is divided into two parts, its stiffness will be twice *i.e.*, $2k$, but now two parts of the spring are in parallel, so $k_e = 4k$.

Then $$\omega_n = \sqrt{\frac{4k}{m_1}} = \sqrt{\frac{4 \times 973}{2.5}} = 39.45 \text{ rad/sec}$$

Natural period $$\frac{2\pi}{\omega_n} = \frac{2\pi}{39.45} = 0.159 \text{ sec.}$$

Example 2.33 *A circular cylinder of mass m and radius r is connected by a spring of stiffness k on an inclined plane as shown in Fig. 2.37. If it is free to roll on the rough surface which is horizontal without slipping, determine its natural frequency.*

(DCRU, Murthal, 2011 ; P.U., Aero 92)

Solution. By applying Newton's second law, the equation of motion can be written as

$$m\ddot{x} = -kx + F_f$$

where F_f = friction force

$$m\ddot{x} + kx - F_f = 0 \quad \text{...(1)}$$

Also $$I\ddot{\theta} = -F_f \cdot r$$

$$\frac{1}{2} mr^2 \ddot{\theta} = -F_f \cdot r$$

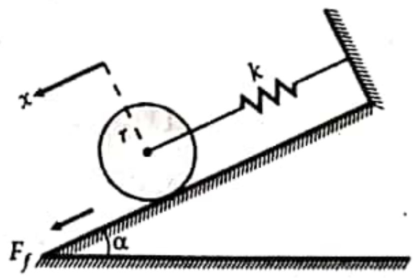


Fig. 2.37

At any moment

$$\therefore \quad F_f = -\frac{1}{2} mr\ddot{\theta} \qquad (x = r\theta)$$

$$= -\frac{1}{2} mr\frac{\ddot{x}}{r} \qquad (\ddot{x} = r\ddot{\theta})$$

$$= -\frac{1}{2} m\ddot{x} \quad \text{...(2)}$$

Equation (1) can be written as

$$m\ddot{x} + kx + \frac{1}{2} m\ddot{x} = 0$$

$$\frac{3}{2} m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{2k}{3m} x = 0$$

So $\quad \omega_n = \sqrt{\frac{2k}{3m}}$ rad/sec

Example 2.34 *Consider the system shown in Fig. 2.38.*

If $k_1 = 2$ *N/mm,* $k_2 = 1.5$ *N/mm,* $k_3 = 3.0$ *N/mm,* $k_4 = k_5 = 1.5$ *N/mm*

Find mass W if the system has a natural frequency of 10 Hz.

(P.U., Aero 91)

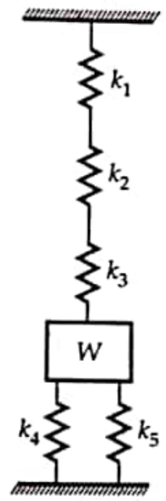


Fig. 2.38

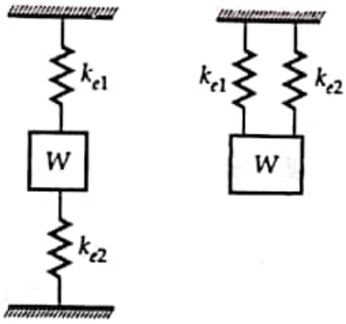


Fig. 2.39

Solution. The springs k_1, k_2 and k_3 are in series, let k_{e1} be their equivalent stiffness

$$\frac{1}{k_{e1}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

$$= \frac{1}{2} + \frac{1}{1.5} + \frac{1}{3} = 1.499$$

$$k_{e1} = 0.667 \text{ N/mm}$$

The two lower springs k_4 and k_5 are connected in parallel so their equivalence k_{e2}

$$k_{e2} = k_4 + k_5$$

$$= .5 + .5 = 1.0 \text{ N/mm}$$

Again k_{e1} and k_{e2} are in parallel, so their equivalence k_e

$$k_e = k_{e1} + k_{e2} = 1.667 \times 10^3 \text{ N/m}$$

$$f_n = 10 \text{ Hz}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_e}{m}}$$

$$f_n^2 = \frac{1}{4\pi^2} \frac{k_e}{m}$$

$$(10)^2 = \frac{1}{4\pi^2} \frac{1.667 \times 1000}{W}$$

$$W = 0.422 \text{ kg}$$

Example 2.25 *Using energy method find the natural frequency of the system shown in Fig. 2.30. The cord may be assumed inextensible in the spring mass pulley system and no slip.*

(PTU and Murthal, 2010)

Given $J = \frac{1}{2} mr^2$ *for pulley.*

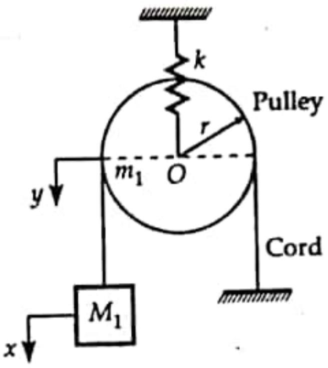


Fig. 2.30

Solution. T = K.E. of mass + K.E. of pulley

$$= \frac{1}{2} M_1 \dot{x}^2 + \frac{1}{2} m\dot{y}^2 + \frac{1}{2} J\dot{\theta}^2$$

For any small displacement θ,

$y = r\theta$ (x = *displacement of mass m at any instant*)

and $y = x/2$ (y = *vertical displacement of pulley centre*)

$$\therefore \quad T = \frac{1}{2} M_1\dot{x}^2 + \frac{1}{2} m(\dot{x}/2)^2 + \frac{1}{2}\cdot\frac{1}{2} mr^2 \frac{\dot{y}^2}{r^2} = \frac{1}{2} M_1\dot{x}^2 + \frac{3}{16} m\dot{x}^2$$

The potential energy is given by

$$V = \text{P.E.} = \frac{1}{2} ky^2 = \frac{1}{2} k\left(\frac{1}{2}x\right)^2 = \frac{1}{8} kx^2$$

The energy of the system is constant.

So $$\frac{d}{dt}(T+V) = 0$$

$$M_1\dot{x}\ddot{x} + \frac{3}{8} m\dot{x}\ddot{x} + \frac{1}{4} kx\dot{x} = 0$$

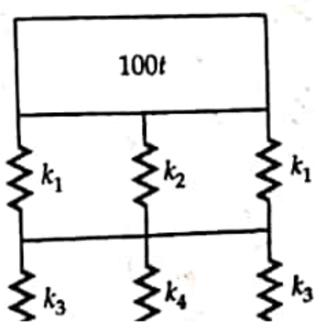
$$M_1\ddot{x} + \frac{3}{8} m\ddot{x} + \frac{1}{4} kx = 0$$

$$\left(M_1 + \frac{3}{8} m\right)\ddot{x} + \frac{1}{4} kx = 0$$

So natural frequency is

$$\omega_n = \sqrt{\frac{1}{4}\frac{k}{\left(M_1 + \frac{3m}{8}\right)}} \text{ rad/sec}$$

$$= \sqrt{\frac{k}{\left(4M_1 + \frac{3m}{2}\right)}} \text{ rad/sec}$$



Example 2.40 *Determine the natural frequency of the mass $m = 15$ kg as shown in Fig. 2.44, assuming that the cords do not stretch and slide over the pulley rim. Assume that the pulley has no mass.* **(P.T.U., 2010)**

Given, $k_1 = 8 \times 10^3$ N/m, $k_2 = 6 \times 10^3$ N/m

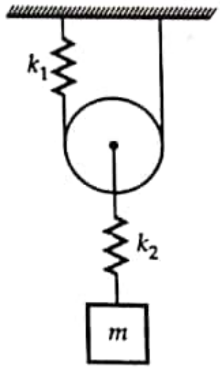


Fig. 2.44

Solution. At any instant the force in spring k_2 is twice to that of k_1 as mass m is acting at the centre of pulley. If we find equivalent stiffness of the system, the natural frequency can be determined.

Let us say P is the force in spring k_2 and so $P/2$ will be in k_1.

Displacement of mass m is given as $x_2 = \dfrac{P}{k_2}$

So extension in spring k_1 is $\quad x_1 = \dfrac{P}{2k_1}$

The mass moves half of x_1 *i.e.*, $x_1/2$

Thus the displacement of mass m due to k_1 is given as

$$\frac{x_1}{2} = x_3 = \frac{P}{4k_1}$$

Total movement of mass m is given as

$$x = \frac{x_1}{2} + x_2 = \frac{P}{4k_1} + \frac{P}{k_2}$$

The equivalent stiffness of the system.

$$k_e = \frac{P}{x} = \frac{P}{\dfrac{P}{4k_1} + \dfrac{P}{k_2}} = \frac{1}{\dfrac{1}{4k_1} + \dfrac{1}{k_2}} = \frac{4k_1k_2}{k_2 + 4k_1}$$

The natural frequency of mass m is given as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_e}{m}} = \frac{1}{2\pi}\sqrt{\frac{4k_1k_2}{(k_2+4k_1)m}} = \frac{1}{\pi}\sqrt{\frac{k_1k_2}{(k_2+4k_1)m}}$$

$$= \frac{1}{\pi}\sqrt{\frac{8\times10^3 \times 6\times10^3}{(6\times10^3 + 4\times8\times10^3)15}} = 2.9 \text{ Hz}$$

EXAMPLE 2.9. *Determine the natural frequency of the spring mass pulley system shown in figure 2.16.*

(P.U., 89)

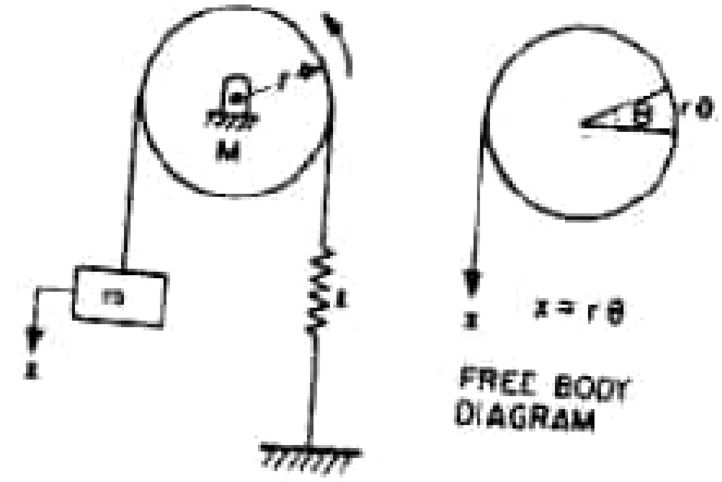


Fig. 2.16. Spring-pulley-mass system

SOLUTION. The total energy T of the system

T = kinetic energy of the mass m + kinetic energy of pulley M + potential energy of spring k

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} kx^2 = \frac{1}{2} mr^2\dot{\theta}^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} kr^2\theta^2 = \text{constant}$$

From free body diagram it is clear that at any moment $x = r\theta$

Moment of inertia of pulley $I = 1/2\ Mr^2$

Differentiating total energy equation with respect to time, we get

$$mr^2\dot{\theta}\ddot{\theta} + I\dot{\theta}\ddot{\theta} + kr^2\theta\dot{\theta} = 0$$

$$mr^2\ddot{\theta} + I\ddot{\theta} + kr^2\theta = 0$$

$$(mr^2 + 1/2\ Mr^2)\ddot{\theta} + kr^2\theta = 0$$

$$\ddot{\theta} + \frac{kr^2}{mr^2 + \frac{1}{2} Mr^2}\theta = 0$$

$$\omega_n = \sqrt{\frac{2kr^2}{2mr^2 + Mr^2}} = \sqrt{\frac{2k}{2m + M}} \text{ rad/sec}$$

EXAMPLE 2.10. *A circular cylinder of mass 4 kg and radius 15 cm. is connected by a spring of stiffness 4000 N/m as shown in figure 2.17. It is free to roll on horizontal rough surface without slipping, determine the natural frequency.*

(P.U. 93, P.U., 99)

Fig. 2.17

SOLUTION. Total energy of the system

$$T = \text{K.E. due to translatory motion} + \text{K.E. due to rotary motion} + \text{P.E. of spring}$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J\dot{\theta}^2 + \frac{1}{2} kx^2$$

$$= \frac{1}{2} mr^2\dot{\theta}^2 + \frac{1}{2} \cdot \frac{1}{2} mr^2\dot{\theta}^2 + \frac{1}{2} kr^2\theta^2 \qquad (\because x = r\theta)$$

$$T = \frac{3}{4} m^2 r^2 \dot{\theta}^2 + \frac{1}{2} kr^2\theta^2 = \text{constant}$$

Differentiating T with respect to time, we get

$$0 = \frac{3}{4} \cdot 2\, mr^2\dot{\theta}\ddot{\theta} + kr^2\theta\dot{\theta} = 0$$

$$\frac{3}{2} mr^2\ddot{\theta} + kr^2\theta = 0$$

$$\omega_n = \sqrt{\frac{kr^2}{3/2\, mr^2}} = \sqrt{\frac{2k}{3m}} \text{ rad/sec}$$

$$f_n = \frac{1}{2\pi} \sqrt{\frac{2 \times 4000}{3 \times 4}} = 4.11 \text{ Hz}$$

Dt. 08.08.2020

ONLINE CLASS-09

CHAPTER-3: Free Damped Vibrations Of Single Degree of Freedom

Damping :

→ It is the dissipation of energy from a vibrating system.

→ When the energy of a vibrating system is gradually dissipated by friction and other resistances the vibrations are said to be damped.

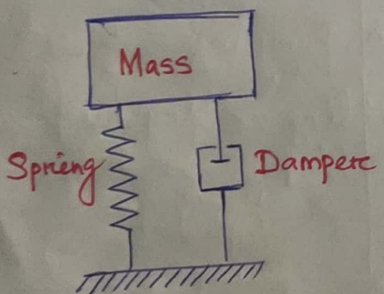
Example- Shock absorbers in bikes

If they are absent, the vibration induced by the road-surface gets passed on to the rider and that will be a rough ride and creating fatigue.

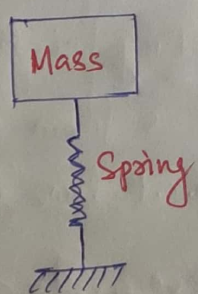
By providing the shock absorbers, the fluid or air or oil present in them acts as a cushion and damps (i.e., diminish) the vibrations from the system.

Damped Vibrations $\xrightarrow[\text{to}]{\text{refers}}$ the reduction of the amplitude of the vibration

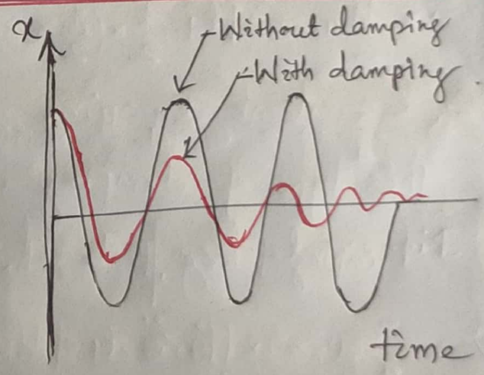
OR the reduction of the duration of the vibration oscillations.



(Fig- Spring-mass-damper System)



(Fig- Spring-Mass System)



(Displacement-time curve)

Types of Dampings :

There are mainly following 5 types dampings used in mechanical systems.

1. Viscous Damping
2. Coulomb Damping / Dry friction Damping
3. Solid or structural Damping or Material / Hysteresis damping
4. Slip or Interfacial damping
5. Magnetic Damping.

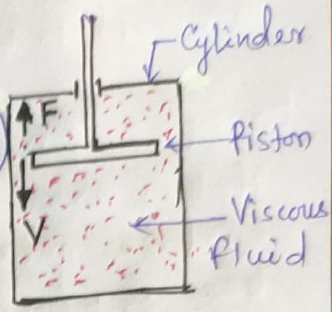
1. Viscous Damping: The damping provided by the fluid resistance (i.e., by viscosity of fluid) is known as viscous damping.

→ It is encountered by bodies moving at moderate speed through fluid medium like air, gas, water or oil.

→ This the most commonly used damping mechanism to reduce the amplitude of vibrations.

→ In case of viscous damping,

Viscous force, $F \propto \dot{x}$ $\quad (\because \dot{x} = \frac{dx}{dt} = \text{velocity})$



Or $\boxed{F = c\dot{x}}$

where 'c' is a constant of proportionality and is known as 'viscous damping co.efficient' [Damper]

$\boxed{c = \frac{F}{\dot{x}}}$, $\frac{N\text{-}s}{m}$

Damping force is opposite to the dir^n of velocity

Applications :

- Door closer
- Damper or dashpot or shock absorbers
- Automobile shock absorbers
- Aircraft carrier docks.

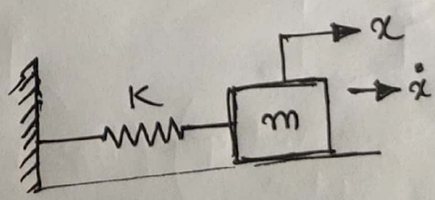
2. Coulomb Damping or Dry friction Damping

This type of damping arises from sliding or dry surfaces.

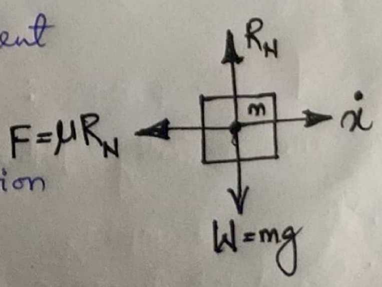
Damping force, $\boxed{F = \mu R_N}$

μ = co.efficient of friction,

R_N = normal reaction

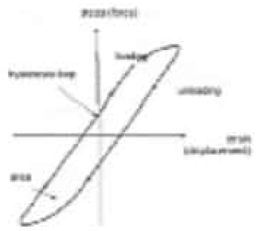


- The damping force, F is independent of velocity of motion.
- Coulomb damping causes the motion to decay linearly.



3.Solid or structural Damping or Hysteretic Damping

This is due to internal friction within the material itself. The damping caused by the friction between the internal planes that slip or slide as the material deforms is called hysteresis (or solid or structural) damping. The stress strain diagram for vibrating body is not straight line but forms hysteresis loop, the area of which represents energy dissipated to molecular friction per cycle per unit volume.



4.Slip or Intrefacial damping

Energy of vibration is dissipated by microscopic slip on the interfaces of machine parts in contact under fluctuating loads. Microscopic slip also occurs on the interface of the machine elements having various types of joints. The amount of damping depends upon the surface roughness of a mating parts, the contact pressure and amplitude of vibration.

5. Magnetic Damping

A phenomenon that has been observed for many years by which vibrating, oscillating or rotating conductors are slowly be brought to rest in the presence of a magnetic field.

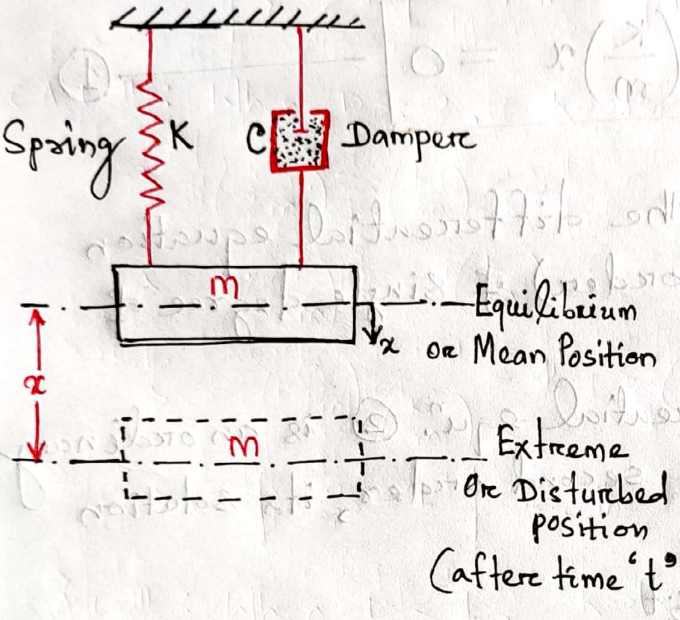
Damping due to eddy currents setup by the movement of a system in a magnetic field.

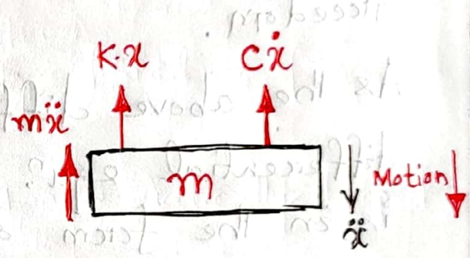
Date- 13-08-2020
ONLINE CLASS-10

Chapter-3 : Free Damped Vibrations (Contd..)

Topic : Differential Equⁿ of Motion of Free Damped Vibration & Its Natural Frequency (ω

Consider a spring-mass-damper vibrating system as shown in the figure. Here the damping provided is viscous damping





[Free Body Diagram]

When the mass (m) is displaced by a distance 'x' from the equilibrium or mean position in time 't', the force acting on the body are,

(i) Inertia force ↑ = − (Accelerating force in the direction of motion)

$= -m\ddot{x}$

(ii) Damping force ↑ = Opposite in the direction of motion

$= -($ damping co-efficient × Velocity $)$

$= -c\dot{x}$

(iii) Spring force ↑ = Opposite in the direction of motion

$= -($ stiffness of spring × displacement $)$

$= -Kx$

For the dynamic equilibrium of the vibrating system, according to D'Alembert Principle,

Inertia force + External force = 0

$$\Rightarrow \quad -m\ddot{x} + (-c\dot{x} - kx) = 0$$

$$\Rightarrow \quad -m\ddot{x} - c\dot{x} - kx = 0$$

$$\Rightarrow \quad \boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Or

$$\boxed{\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0} \quad \text{——— (1)}$$

The above equⁿ (1) is the differential equation of motion (of 2nd order) of single degree of freedom.

As the above differential equⁿ (2) is an ordinary differential equⁿ of second order, its solution is in the form of

$$x = Ae^{\alpha_1 t} + B.e^{\alpha_2 t} \quad \text{——— (2)}$$

where A and B are the arbitrary constants which are obtained from the initial and boundary conditions and α_1 & α_2 are the roots of the auxiliary equⁿ of equⁿ (1),

$$\alpha^2 + \left(\frac{c}{m}\right)\alpha + \left(\frac{k}{m}\right) = 0 \quad \text{——— (3)}$$

$$\therefore \alpha = \frac{-\left(\frac{c}{m}\right) \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\times1\times\frac{k}{m}}}{2\times1}$$

$$= -\left(\frac{c}{2m}\right) \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

So,

$$\left.\begin{aligned} \alpha_1 &= -\left(\frac{c}{2m}\right) + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\ \alpha_2 &= -\left(\frac{c}{2m}\right) - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \end{aligned}\right\} \text{two roots of auxiliary equ}^n \text{ (3)}$$

In the above quadratic equation,

Discriminant, $D = b^2 - 4ac = \left(\frac{c}{2m}\right)^2 - \left(\frac{K}{m}\right)$

CASE-I : When $D > 0$, Roots are real

$\Rightarrow \left(\frac{c}{2m}\right)^2 - \left(\frac{K}{m}\right) > 0$

$\Rightarrow \left(\frac{c}{2m}\right)^2 > \frac{K}{m}$

$\Rightarrow \left(\frac{c}{2m}\right)^2 \Big/ \left(\frac{K}{m}\right) > 1 \Rightarrow \xi^2 > 1$ $\left[\because \frac{\left(\frac{c}{2m}\right)^2}{\left(\frac{K}{m}\right)} = \xi^2\right]$

$\Rightarrow \boxed{\xi > 1}$ i.e., The system is overdamped.

CASE-II : When $D = 0$, Roots are real and equal

$\Rightarrow \left(\frac{c}{2m}\right)^2 - \left(\frac{K}{m}\right) = 0$

$\Rightarrow \left(\frac{c}{2m}\right)^2 = \left(\frac{K}{m}\right)$

$\Rightarrow \left(\frac{c}{2m}\right)^2 \Big/ \left(\frac{K}{m}\right) = 1 \Rightarrow \xi^2 = 1$ $\left[\because \frac{\left(\frac{c}{2m}\right)^2}{\left(\frac{K}{m}\right)} = \xi\right]$

$\Rightarrow \boxed{\xi = 1}$ i.e., The system is critically damped

CASE-III : When $D < 0$, Roots are imaginary

$\Rightarrow \left(\frac{c}{2m}\right)^2 - \left(\frac{K}{m}\right) < 0$

$\Rightarrow \left(\frac{c}{2m}\right)^2 < \left(\frac{K}{m}\right)$

$\Rightarrow \left(\frac{c}{2m}\right)^2 \Big/ \left(\frac{K}{m}\right) < 1 \Rightarrow \xi^2 < 1$ $\left[\because \frac{\left(\frac{c}{2m}\right)^2}{\left(\frac{K}{m}\right)} = \xi^2\right]$

$\Rightarrow \boxed{\xi < 1}$ i.e., The system is under damped.

Imp. Terms :

Degree of dampness $= \frac{\left(\frac{c}{2m}\right)^2}{\left(\frac{K}{m}\right)} = \frac{c^2}{4Km}$

Damping factor or Damping factor (ξ) :

Read as $\xi \leftarrow$ zeta

$$\xi = \sqrt{\text{Degree of dampness}} = \frac{c}{2\sqrt{Km}}$$

Damping co-efficient (c) : $c = 2\xi\sqrt{Km} = 2\xi m\omega_n = 2\xi \frac{K}{\omega_n}$

Critically damping co.efficient (C_c) : When the damping ratio, $\xi = 1$ then the damping is known as critical damping and the corresponding value of damping co.efficient is known as C_c.

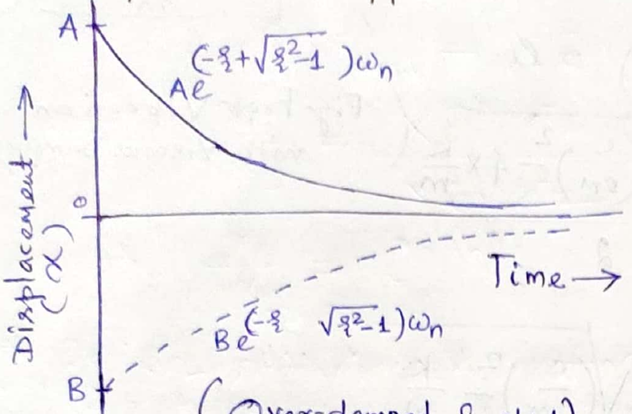
$$C_c = 2\sqrt{Km} = 2m\omega_n = \frac{2K}{\omega_n}$$

* Also $$\boxed{\xi = \frac{C}{C_c} = \frac{\text{Actual damping co.efficient}}{\text{Critical damping co.efficient}}}$$

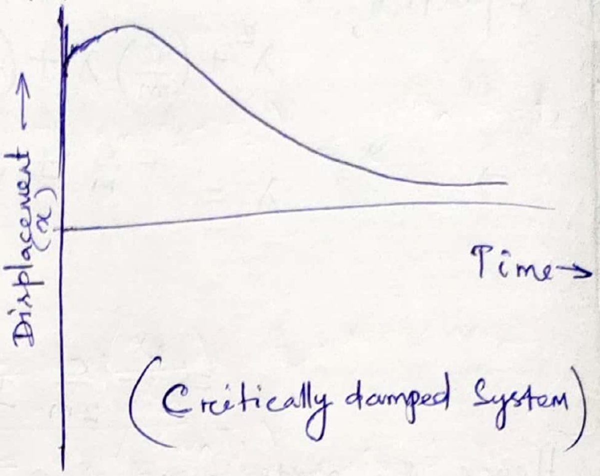
CASE-I: $\xi > 1$, i.e., the system is over-damped

$$\boxed{x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}} = Ae^{(-\xi+\sqrt{\xi^2-1})\omega_n t} + Be^{(-\xi-\sqrt{\xi^2-1})\omega_n t} \quad (1)$$

This is an equation of an aperiodic motion, i.e., the system can not vibrate due to over-damping. The magnitude of the resultant displacement approaches zero with time.



(Over-damped System)



(Critically damped System)

CASE-II: $\xi = 1$, i.e., the damping is critical

(Roots of the auxiliary equation are equal and each being = to $-\omega_n$)

$$\boxed{x = (A+B)e^{-\omega_n t}} \quad (2)$$

Since $e^{-\omega_n t}$ approaches ZERO as $t \to \infty$, the motion is aperiodic. The displacement will be approaching to ZERO with time.

In a critically damped system, the displaced/disturbed mass returns to the position of rest in the shortest possible time without oscillations/vibrations.

Due to this reason, Large guns, tankers, missile launchers are designed with critically damped so that they return to their original position/mean position (after recoiling because of firing) in the minimum possible time.

CASE-III: $\xi < 1$, i.e., the system is under-damped

Under this condition, the roots α_1 & α_2 are complex conjugate

$$\alpha_1 = (-\xi + i\sqrt{1-\xi^2})\omega_n$$

& $$\alpha_2 = (-\xi - i\sqrt{1-\xi^2})\omega_n$$

So, the displacement $x = Ae^{\alpha_1 t} + Be^{\alpha_2 t} = Ae^{(-\xi+i\sqrt{1-\xi^2})\omega_n t} + Be^{(-\xi-i\sqrt{\xi^2-1})\omega_n t}$

$$= e^{-\xi\omega_n t}\left(Ae^{i\sqrt{1-\xi^2}\,\omega_n t} + Be^{-i\sqrt{1-\xi^2}\,\omega_n t}\right)$$

After few lines of simplification, $x = Xe^{-\xi\omega_n t}\sin\left(\sqrt{1-\xi^2}\,\omega_n t + \phi\right)$

where X = constant.

Or $\boxed{x = Xe^{-\xi\omega_n t} \cdot \sin(\omega_d t + \phi)}$ ——— (3)

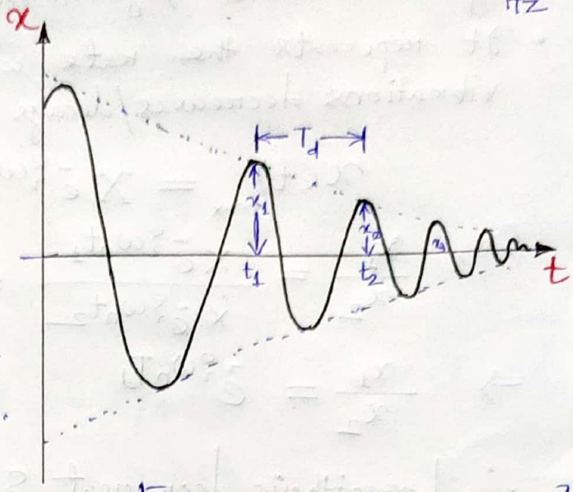
Here ω_d = damped frequency, rad/s

$\Rightarrow \boxed{\omega_d = \sqrt{1-\xi^2}\,\omega_n}$, rad/s

$\boxed{f_d = \frac{\omega_d}{2\pi} = \frac{\sqrt{1-\xi^2}\,\omega_n}{2\pi} = \sqrt{1-\xi^2}\cdot f_n}$ Hz

The above equn (3) represents an equation of simple harmonic motion (SHM) with amplitude $Xe^{-\xi\omega_n t}$ and damped frequency, ω_d $(=\sqrt{1-\xi^2}\,\omega_n)$.

The factor $Xe^{-\xi\omega_n t}$ goes on decreasing exponentially with time 't'. Hence, in case of under-damped system, the vibrating sys. vibrates with decreasing amplitude.



[Fig- Under-damped System]

Damped Frequency, $\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$, rad/s

$\omega_d < \omega_n$ $\boxed{f_n = f_n\sqrt{1-\xi^2}}$, Hz $\left(\because f_n = \frac{\omega_d}{2\pi} = \frac{\omega_n\sqrt{1-\xi^2}}{2\pi} = f_n\sqrt{1-\xi^2}\right)$

Damped Time Period, $T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\xi^2}\,\omega_n} = \frac{T}{\sqrt{1-\xi^2}}$, sec.

The amplitude $Xe^{-\xi\omega_n t}$ decreases with time (t) and finally $e^{-\infty} \simeq 0$

$\sin(\omega_d t + \phi)$ represents repetition of motion. Thus, the resultant motion is oscillatory with decreasing amplitudes and having a frequency, ω_d (or f_d).

In case of under-damped vibrations, the ratio of successive amplitudes remains same, i.e., $\frac{x_1}{x_2} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \cdots = \frac{x_n}{x_{n+1}}$ = constant

Here, the amplitudes $(x_1, x_2, \ldots,)$ of successive vibrations of an under-damped system are Geometric Progression (G.P.) and their ratios (i.e., $\frac{x_1}{x_2}$ or $\frac{x_2}{x_3}$ ....) represent as a common ratio in a G.P. series.

Logarithmic Decrement (δ):

- The ratio of two successive amplitude of vibrations/oscillations in an under-damped system is constant.
- The natural logarithm of any two successive amplitude ratio on the same side of mean position in an under-damped system is called logarithmic decrement (δ). $\boxed{\delta = \ln \frac{x_1}{x_2}}$
- It repesents the rate which the amplitude of under-damped vibrations decreases/decays with time (t).

$$x(t)_{max} = X e^{-\xi \omega_n t}$$

Or,
$$\frac{x_1}{x_2} = \frac{X e^{-\xi\omega_n t_1}}{X e^{-\xi\omega_n t_2}} = e^{-\xi\omega_n(t_1-t_2)} = e^{\xi\omega_n(t_2-t_1)}$$

$$\Rightarrow \frac{x_1}{x_2} = e^{\xi\omega_n T_d}$$

$\therefore$ Logarithmic decrement,
$$\delta = \ln\left(\frac{x_1}{x_2}\right)$$
$$= \ln\left(e^{\xi\omega_n T_d}\right)$$
$$= \xi\omega_n T_d \times \ln e$$
$$= \xi\omega_n \times \frac{2\pi}{\sqrt{1-\xi^2}\cdot\omega_n}$$
$$\Rightarrow \boxed{\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}}$$

If the cycle executes 'n' cycles/oscillations, the logarithmic decrement, $\boxed{\delta = \frac{1}{n} \ln \frac{x_1}{x_{n+1}}}$

Proof: Logarithmic decrement, $\delta = \ln \frac{x_1}{x_2}$

$$\Rightarrow \frac{x_1}{x_2} = e^{\delta}$$

We can write
$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \cdots\cdots = \frac{x_n}{x_{n+1}} = e^{\delta}$$

$$\therefore \left(\frac{x_1}{x_2}\right)\left(\frac{x_2}{x_3}\right)\cdots\cdots\left(\frac{x_{n-1}}{x_n}\right)\times\left(\frac{x_n}{x_{n+1}}\right) = e^{n\delta}$$

$$\Rightarrow \frac{x_1}{x_{n+1}} = e^{n\delta}$$

$$\ln\left(\frac{x_1}{x_{n+1}}\right) = \ln e^{n\delta} = n\delta \times \ln e$$

$$\Rightarrow \boxed{\delta = \frac{1}{n}\ln\frac{x_1}{x_{n+1}}}$$

Problem-1 : A vibrating system is defined by the following parameters

$m = 3$ kg , $K = 100$ N/m , $c = 3$ N-sec/m

Determine : (a) the damping factor (b) the natural frequency of damped vibration

(c) Logarithmic decrement (d) the ratio of two consecutive amplitudes

(e) the number of cycles after which the original amplitude is reduced to 20 percent.

Solution :- (a) $\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{100}{3}} = 5.773$ rad/sec

Damping factor, $\xi = \frac{c}{2m\omega_n} = \frac{3}{2 \times 3 \times 5.773} = 0.086$

(b) $\omega_d = \omega_n\sqrt{1-\xi^2} = 5.773\sqrt{1-(0.086)^2} = 5.751$ rad/s

(c) $\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi \times 0.086}{\sqrt{1-(0.086)^2}} = 0.542$

(d) Also we know, $\delta = \ln\frac{x_1}{x_2}$

$\Rightarrow \frac{x_1}{x_2} = e^{\delta} = e^{0.542} = 1.719$

(e) It is given that, $X_{n+1} = 0.20\,X_1$

So, $\delta = \frac{1}{n}\ln\frac{X_1}{X_{n+1}}$

$\Rightarrow n = \frac{1}{\delta}\ln\frac{X_1}{X_{n+1}} = \frac{1}{0.542} \times \ln\left(\frac{X_1}{0.20X_1}\right) = \frac{1}{0.542} \times \ln(5)$

$\Rightarrow n = 2.96$ cycles **Ans**